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Monadic Basic Algebras*

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Abstract

The concept of monadic MV-algebra was recently introduced by A. Di Nola and R. Grigolia as an algebraic formalization of the many-valued predicate calculus described formerly by J. D. Rutledge [9]. This was also genaralized by J. Rachůnek and F. Švrček for commutative residuated ℓ-monoids since MV-algebras form a particular case of this structure. Basic algebras serve as a tool for the investigations of much more wide class of non-classical logics (including MV-algebras, orthomodular lattices and their generalizations). This motivates us to introduce the monadic basic algebra as a common generalization of the mentioned structures.

Key words: Basic algebra; monadic basic algebra; existential quantifier; universal quantifier; lattice with section antitone involution.

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Having an MV-algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, one can derive the structure of bounded distributive lattice $\mathcal{L}(A) = (A; \vee, \wedge, 0, 1)$ where $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Moreover, to any element $a \in A$ one can assign an antitone involution $x \mapsto x^a$ on the interval $[a, 1]$ in $\mathcal{L}(A)$ given by $x^a = \neg x \oplus a$ (for $x \in [a, 1]$). Hence, $\mathcal{L}(A)$ is a lattice equipped by a set $({}^a)_{a \in A}$ of partial unary operations defined on the so-called *sections* where for each $x \in [a, 1]$ we have $x^{aa} = x$ and for $x, y \in [a, 1]$ with $x \leq y$ we have $y^a \leq x^a$ (see e.g. [3] for

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details). Such an enriched lattice (not necessarily distributive) is denoted by $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ and is called a *lattice with section antitone involutions.*

Although this structure plays a crucial role in some formalizations of non-classical logics, it can be difficult to deal with since it is not a total algebra and, moreover, its similarity type depends on the cardinality of its elements. To improve this discrepancy, the following concept was introduced. Let us only note that the following axiom system (BA1)–(BA4) was recently involved in [6] as a simplification of the previous one (see e.g. [1, 2, 5]).

Definition 1 By a *basic algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$ (double negation);

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

In what follows we will denote $\neg 0$ by 1 (as it is usual for MV-algebras). It is plain to show that every basic algebra satisfies also the identities $\neg 1 = 0$, $0 \oplus x = x$ and $\neg x \oplus x = 1$, see e.g. [4, 6].

As promised above, we can get the mutual relationship between lattices with section antitone involutions and basic algebras. For the proof, see e.g. [1] or [5].

Proposition 1 *(a) Let* $\mathcal{L} = (L; \vee, \wedge, (^a)_{a\in L}, 0, 1)$ *be a lattice with section antitone involutions. Then the assigned algebra* $\mathcal{A}(L) = (L; \oplus, \neg, 0)$*, where*

$$x \oplus y = (x^0 \vee y)^y \quad \text{and} \quad \neg x = x^0$$

is a basic algebra.

(b) Conversely, given a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$*, we can assign a bounded lattice with section antitone involutions* $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a\in A}, 0, 1)$*, where* $1 = \neg 0$*,*

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \qquad x \wedge y = \neg(\neg x \vee \neg y)$$

and for each $a \in A$*, the mapping* $x \mapsto x^a = \neg x \oplus a$ *is an antitone involution on the principal filter* $[a, 1]$*, where the order is given by*

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

(c) The assignments are in a one-to-one correspondence, i.e. $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$ *and* $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$*.*

Hence, when investigating basic algebras, we can switch to lattices with section antitone involutions whenever it is useful.

The lattice $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a\in A}, 0, 1)$ will be referred as an *assigned lattice* of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ and the order $\leq$ of $\mathcal{L}(A)$ as the *induced order* of $\mathcal{A}$.

Definition 2 By a *monadic basic algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ of type $(2, 1, 1, 0)$ where $(A; \oplus, \neg, 0)$ is a basic algebra and the unary operation $\exists$ satisfies the following identities

(E1) $x \leq \exists x$;

(E2) $\exists(x \vee y) = \exists x \vee \exists y$;

(E3) $\exists(\neg\exists x) = \neg\exists x$;

(E4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$.

The mapping $\exists\colon A \to A$ is called an *existential quantifier* on $\mathcal{A}$. By a *strict monadic basic algebra* will be called a monadic basic algebra satisfying the identity

(E5) $\exists(x \oplus x) = \exists x \oplus \exists x$.

Lemma 1 *Let $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ be a monadic basic algebra. Then the following conditions are satisfied:*

(i) $\exists 1 = 1$*;*

(ii) $\exists 0 = 0$*;*

(iii) $\exists\exists x = \exists x$*;*

(iv) $x \leq \exists y$ *if and only if* $\exists x \leq \exists y$*;*

(v) if $x \leq y$ *then* $\exists x \leq \exists y$*;*

(vi) $\neg\exists x \leq \exists(\neg x)$*;*

Proof Let x, y be arbitrary elements of $\mathcal{A}$.
(i): By (E1), $1 \leq \exists 1$, thus $\exists 1 = 1$ as 1 is the greatest element of $\mathcal{A}$.
(ii): By (i) and (E3), $0 = \neg 1 = \neg\exists 1 = \exists(\neg\exists 1) = \exists(\neg 1) = \exists 0$.
(iii): By (ii) and (E4), $\exists\exists x = \exists(\exists x \oplus 0) = \exists(\exists x \oplus \exists 0) = \exists x \oplus \exists 0 = \exists x \oplus 0 = \exists x$.
(iv): If $\exists x \leq \exists y$ then by (E1) also $x \leq \exists y$. On the other hand using (iii) and (E2), if $x \leq \exists y$ then $\exists y = \exists\exists y = \exists(x \vee \exists y) = \exists x \vee \exists\exists y = \exists x \vee \exists y$. Thus $\exists y = \exists x \vee \exists y$, and therefore $\exists x \leq \exists y$.
(v): Let $x \leq y$. Then, by (E2), we obtain $\exists y = \exists(x \vee y) = \exists x \vee \exists y$, and hence $\exists x \leq \exists y$.
(vi): Since $x \leq \exists x$ and hence $\neg x \geq \neg\exists x$, we conclude $\exists(\neg x) \geq \neg x \geq \neg\exists x$. □

In what follows let $\exists$ be a fixed existential quantifier defined on a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. By means of $\exists$, a unary operation $\forall$ can be defined on $\mathcal{A}$ by the rule

$$\forall x := \neg(\exists\neg x). \tag{1}$$

Lemma 2 *Let $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ be a monadic basic algebra and $\forall$ is defined by (R). Then the following conditions are satisfied*

(A1) $\forall x \leq x$*;*

(A2) $\forall(x \wedge y) = \forall x \wedge \forall y$*;*

(A3) $\forall(\neg\forall x) = \neg\forall x$;

(A4) $\forall(\forall x \odot \forall y) = \forall x \odot \forall y$, *where* $x \odot y = \neg(\neg x \oplus \neg y)$.

If, moreover, $\mathcal{A}$ is a strict monadic basic algebra, then it satisfies also

(A5) $\forall(x \odot x) = \forall x \odot \forall x$.

Proof By (E1), $\neg x \leq \exists\neg x$ thus $x = \neg\neg x \geq \neg(\exists\neg x) = \forall x$ proving (A1). To prove (A2), we use (E2) and the De Morgan laws:

$$\forall(x \wedge y) = \forall(\neg(\neg x \vee \neg y)) = \neg\exists(\neg x \vee \neg y)$$

$$= \neg((\exists\neg x) \vee (\exists\neg y)) = \neg(\exists\neg x) \wedge \neg(\exists\neg y) = \forall x \wedge \forall y.$$

Prove (A3): $\forall(\neg\forall x) = \neg\exists(\neg\neg\forall x) = \neg\exists(\neg(\exists\neg x)) = \neg\neg(\exists\neg x) = \neg\forall x$ by (E3). For (A4) we compute by (E4)

$$\forall(\forall x \odot \forall y) = \neg(\exists\neg(\neg(\exists\neg x) \odot \neg(\exists\neg y))) = \neg\exists(\exists\neg x \oplus \exists\neg y)$$

$$= \neg(\exists\neg x \oplus \exists\neg y) = \neg(\neg\neg(\exists\neg x) \oplus \neg\neg(\exists\neg y)) = \forall x \odot \forall y.$$

Assume that $\exists$ satisfies also (E5). Then

$$\forall(x \odot x) = \neg(\exists\neg(x \odot x)) = \neg(\exists\neg(\neg\neg x \odot \neg\neg x))$$

$$\neg(\exists(\neg x \oplus \neg x)) = \neg(\exists\neg x \oplus \exists\neg x) = (\neg(\exists\neg x)) \odot (\neg(\exists\neg x)) = \forall x \odot \forall x.$$

□

A unary operation $\forall\colon A \to A$ on a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfying (A1)–(A4) will be called a *universal quantifier*.

It is a routine way to prove also the converse:

Lemma 3 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $\forall$ be a universal quantifier on $\mathcal{A}$. Define*

$$\exists x := \neg(\forall\neg x).$$

Then $\mathcal{A}_\exists = (A; \oplus, \neg, \exists, 0)$ is a monadic basic algebra. Moreover, if it satisfies also (A5) then $\mathcal{A}_\exists$ is a strict monadic basic algebra.

Remark 1 Let $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ be a monadic basic algebra. Then $\exists$ is a closure operator and $\forall$ is an interior operator on the poset $(A; \leq)$, where the relation $\leq$ is the induced order on $\mathcal{A}$.

In what follows, we are going to prove a connection between monadic basic algebras and enriched lattices with section antitone involutions similarly as it was done for basic algebras in the Proposition. For this, let us recall some concepts.

For an algebra $\mathcal{A} = (A; F)$, by a *retraction* is meant an idempotent endomorphism h of $\mathcal{A}$, i.e. an endomorphism satisfying $h(h(x)) = h(x)$ for every $x \in A$. It is well-known that if h is a retraction of $\mathcal{A}$ then its image $\mathcal{A}_0 = h(A)$ is a subalgebra of $\mathcal{A}$, the so-called *retract of* $\mathcal{A}$.

In particular, if $\mathcal{S} = (S; \vee, 0)$ is a join-semilattice with 0, by a *retraction* is meant a self-mapping e of S satisfying (e1) $e(x \vee y) = e(x) \vee e(y), \quad e(0) = 0,$

(e2) $e(e(x)) = e(x).$

This retraction is called *extensive* if it satisfies also

(e3) $x \leq e(x).$

Example 1 Consider the bounded join-semilattice $\mathcal{S} = (A; \vee, 0, 1)$, where $A = \{0, a, b, 1\}$, depicted in Fig. 1.

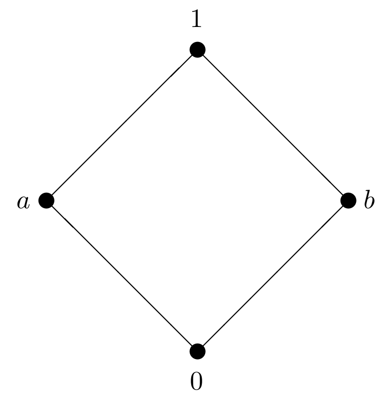

Fig. 1

Define $e\colon A \to A$ as follows

$$e(0) = 0, \quad e(a) = 1, \quad e(b) = b, \quad e(1) = 1.$$

Then e is an extensive retraction of $\mathcal{S}$ and the retract $S_0 = e(S)$ is the chain $\{0, b, 1\}$. Remark that the semilattice $\mathcal{S}$ can be considered also as a lattice but this e is not a lattice retraction since

$$e(a \wedge b) = e(0) = 0 \neq b = 1 \wedge b = e(a) \wedge e(b).$$

Now, let $\mathcal{L} = (L; \vee, \wedge, ({}^a)_{a\in L}, 0, 1)$ be a lattice with section antitone involutions. A mapping $e\colon L \to L$ will be called an **e**-*retraction* if it is an extensive retraction of the join-semilattice reduct $(L; \vee, 0)$ satisfying one more condition

(e4) $e(e(x)^{e(y)}) = e(x)^{e(y)}$ for every pair $y \leq x$.

Let us note that $y \leq x$ implies $e(y) \leq e(x)$ just by (e1).

If e is an e-retraction on a lattice $\mathcal{L} = (L; \vee, \wedge, ({}^a)_{a\in L}, 0, 1)$ with section antitone involutions then the enriched structure $\mathcal{L}_e = (L; \vee, \wedge, ({}^a)_{a\in L}, e, 0, 1)$ will be called a *monadic lattice*.

We are going to prove

Theorem 1 *Let* $\mathcal{L}_e = (L; \vee, \wedge, ({}^a)_{a\in L}, e, 0, 1)$ *be a monadic lattice and* $\mathcal{A}_e(L) = (L; \oplus, \neg, e, 0)$ *an algebra such that* $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ *is a basic algebra assigned to the reduct* $\mathcal{L} = (L; \vee, \wedge, ({}^a)_{a\in L}, 0, 1)$. *Then* $\mathcal{A}_e(L)$ *is a monadic basic algebra.*

Proof We need only to show that $\mathcal{A}_e(L)$ satisfies the conditions (E3) and (E4) from Definition 2. To prove (E3) we compute:

$$e(\neg e(x)) = e(e(x)^0) \overset{(e1)}{=} e(e(x)^{e(0)}) \overset{(e4)}{=} e(x)^{e(0)} \overset{(e1)}{=} e(x)^0 = \neg e(x)$$

Further, we check the following identity

$$e(\neg e(x) \vee e(y)) = \neg e(x) \vee e(y). \tag{2}$$

For this, we compute

$$e(\neg e(x) \vee e(y)) \overset{(E3)}{=} e(e(\neg e(x)) \vee e(y))$$
$$\overset{(e1),(e2)}{=} e(\neg e(x)) \vee e(y) \overset{(E3)}{=} \neg e(x) \vee e(y).$$

Now, we are ready to prove (E4):

$$e(x) \oplus e(y) = (\neg e(x) \vee e(y))^{e(y)} \overset{(A)}{=} (e(\neg e(x) \vee e(y)))^{e(y)}$$
$$\overset{(e4)}{=} e((e(\neg e(x) \vee e(y)))^{e(y)}) \overset{(A)}{=} e((\neg e(x) \vee e(y))^{e(y)}) = e(e(x) \oplus e(y)).$$

□

We can prove also the converse.

Theorem 2 *Let $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ be a monadic basic algebra, let $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a\in A}, 0, 1)$ be the assigned lattice of the reduct $(A; \oplus, \neg, 0)$. Then $\mathcal{L}_\exists(A) = (A; \vee, \wedge, (^a)_{a\in A}, \exists, 0, 1)$ is a monadic lattice.*

Proof We prove that the mapping $e\colon x \to \exists x$ is an e-retraction of $\mathcal{L}_\exists(A)$. Trivially, we have: $e(x \vee y) = \exists(x \vee y) = \exists x \vee \exists y = e(x) \vee e(y)$ and $e(0) = \exists 0 = 0$. Further, $e(e(x)) = \exists\exists x = \exists x = e(x)$ by (iii) of Lemma 1 and $x \leq e(x) = \exists x$ by (E1). We prove (e4): Since $x^y = \neg x \oplus y$ (for $x \in [y, 1]$), we have

$$e(e(x)^{e(y)}) = \exists((\exists x)^{(\exists y)}) = \exists((\neg\exists x) \oplus \exists y) \overset{(E3)}{=} \exists((\exists(\neg\exists x)) \oplus \exists y)$$
$$\overset{(E4)}{=} (\exists(\neg\exists x)) \oplus \exists y \overset{(E3)}{=} (\neg\exists x) \oplus \exists y = (\exists x)^{(\exists y)} = e(x)^{e(y)}.$$

□

Remark 2 If $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ is a strict monadic basic algebra then the assigned monadic lattice $\mathcal{L}(A)$ satisfies the condition

(e5) $e((x^0 \vee x)^x) = (e(x)^0 \vee e(x))^{e(x)}$

(where $e(x)$ stands for $\exists x$ in $\mathcal{A}$) and vice versa, if a monadic lattice $\mathcal{L}$ satisfies (e5) then the assigned monadic basic algebra $\mathcal{A}(L)$ is strict.

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. It is plain to check that the identity mapping $id(x) = x$ is an existential quantifier on $\mathcal{A}$. Moreover, define a mapping $j\colon A \to A$ as follows

$$j(0) = 0 \quad \text{and} \quad j(x) = 1 \quad \text{for } x \neq 0.$$

Then also j is an existential quantifier on $\mathcal{A}$. Hence, by Theorem 2, id and j are e-retractions on the assigned lattice $\mathcal{L}(A)$.

Example 2 For a basic algebra $\mathcal{H} = (H; \oplus, \neg, 0)$ with $H = \{0, a, b, 1\}$, where $\neg 0 = 1$, $\neg a = a$, $\neg b = b$, $\neg 1 = 0$, the assigned lattice is depicted in Fig. 2 (the antitone involutions in at most two-elements sections are determined uniquely).

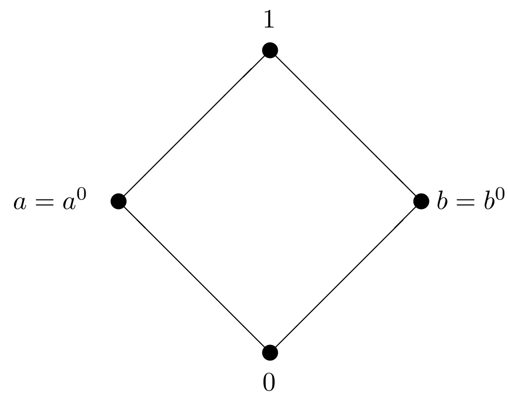

Fig. 2

There are four of e-retractions, namely id, j and h_1, h_2 defined by

$$h_1(0) = 0, \quad h_1(1) = 1, \quad h_1(a) = a, \quad h_1(b) = 1$$

and

$$h_2(0) = 0, \quad h_2(1) = 1, \quad h_2(a) = 1, \quad h_2(b) = b.$$

In what follows, we can borrow the following concept of relatively complete subalgebra, defined for MV-algebras in [7] and for residuated ℓ-monoids in [8]:

Definition 3 A subalgebra B of a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called *relatively complete* if for every $a \in A$ the set $\{b \in B;\, a \leq b\}$ has the least element. Further, a relative complete subalgebra B is called *m-relatively complete* if

$$\begin{aligned}&\text{for all } a \in A \text{ for all } b \in B\colon b \geq a \oplus a \text{ implies}\\ &\quad \text{that there exists } v \in B\colon v \geq a \text{ and } b \geq v \oplus v.\end{aligned} \tag{3}$$

Theorem 3 *Let $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ be a monadic basic algebra and $A_0 = \{\exists x;\, x \in A\}$. Then $\mathcal{A}_0$ is a relatively complete subalgebra of $\mathcal{A}$. If, moreover, $\mathcal{A}$ is a strict monadic basic algebra then $\mathcal{A}_0$ is an m-relatively complete subalgebra of $\mathcal{A}$.*

Proof Due to (E3), (E4) and (ii), (iii) of Lemma 1, $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$. Let $a \in A$ and $B_a = \{b \in A_0;\, a \le b\}$. Then $\exists a \in B_a$ and for any $b \in B_a$ we have $b = \exists d$ for some $d \in A$. Hence, $\exists a \le \exists\exists d = \exists d = b$, thus $\exists a$ is the least element of B_a. Hence, $\mathcal{A}_0$ is a relatively complete subalgebra of $\mathcal{A}$. Assume that $\mathcal{A}$ is a strict monadic basic algebra. Let $a \in A$, $b \in A_0$ and $b \ge a \oplus a$. Then for $v = \exists a$ we have $v \ge a$ due to (E1) and, due to (E5), $b = \exists b \ge \exists(a \oplus a) = \exists a \oplus \exists a = v \oplus v$ proving (C). □

We have shown that any existential quantifier $\exists$ on a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ induces a relatively complete subalgebra $\mathcal{A}_0 = \exists A$ of $\mathcal{A}$. Also conversely, every relatively complete subalgebra of $\mathcal{A}$ gives rise to an existential quantifier.

We say that a basic algebra $\mathcal{A}$ is *$\oplus$-monotonous* if $x \ge y$ implies $x \oplus x \ge y \oplus y$. Let us note that e.g. every MV-algebra or an effect algebra satisfies this condition.

Theorem 4 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $\mathcal{A}_0$ its relatively complete subalgebra. For any $a \in A$, define $\exists a = \inf\{b \in A_0;\, a \le b\}$. Then $\mathcal{A}_\exists = (A; \oplus, \neg, \exists, 0)$ is a monadic basic algebra. If, moreover, $\mathcal{A}$ is $\oplus$-monotonous and $\mathcal{A}_0$ is an m-relatively complete subalgebra of $\mathcal{A}$ then $\mathcal{A}_\exists = (A; \oplus, \neg, \exists, 0)$ is a strict monadic basic algebra.*

Proof It is evident that $x \le \inf\{b \in A_0;\, x \le b\} = \exists x$ and that $x \le y$ implies $\exists x \le \exists y$, i.e. also $\exists(x \vee y) \ge \exists x \vee \exists y$. Since $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ and $\exists x, \exists y \in A_0$, also $\exists x \vee \exists y \in A_0$ and $x \le \exists x$, $y \le \exists y$ thus also $x \vee y \le \exists x \vee \exists y$. Hence, $\exists x \vee \exists y \in \{b \in A_0;\, x \vee y \le b\} = B_{x \vee y}$, i.e. $\exists(x \vee y) = \inf B_{x \vee y} \le \exists x \vee \exists y$.

Evidently, $\exists x = x$ for any $x \in A_0$. Since $\exists x \in A_0$ for each $x \in A$ and $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$, it yields also $\neg\exists x \in A_0$ and hence $\exists(\neg\exists x) = \neg\exists x$. We obtain $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$ in a similar way.

Altogether, $\mathcal{A} = (A; \oplus, \neg, \exists, 0)$ is a monadic basic algebra.

Assume now that $\mathcal{A}_0$ is an m-relatively complete subalgebra of $\mathcal{A}$. Let $x \in A$ and denote by $D = \{b \in A_0;\, x \le b\}$. Then $\exists(x \oplus x) \ge x \oplus x$ as shown above and, by the condition (C), there exists a $v \in D$ with $\exists(x \oplus x) \ge v \oplus v$. Since $v \in D$ and $\exists x = \inf D$, thus $\exists(x \oplus x) \ge v \oplus v \ge \exists x \oplus \exists x$ by $\oplus$-monotonicity. Conversely, $\exists x \ge x$ yields $\exists x \oplus \exists x \ge x \oplus x$ by $\oplus$-monotonicity of $\mathcal{A}$ and, by (E4),

$$\exists x \oplus \exists x = \exists(\exists x \oplus \exists x) \ge \exists(x \oplus x).$$

□

Let $\mathcal{L}_i$ $(i \in I)$ be bounded lattices (or semilattices). By a *horizontal sum* is meant a lattice (semilattice) $\mathcal{L}$ which is a union of $\mathcal{L}_i$ $(i \in I)$ such that

$$\mathcal{L}_i \cap \mathcal{L}_j = \{0, 1\} \quad \text{for} \quad i \ne j.$$

Let $\mathcal{A}_i$ $(i \in I)$ be basic algebras. By a horizontal sum of $\mathcal{A}_i$ is meant a basic algebra $\mathcal{A}$ assigned to the lattice $\mathcal{L}$ which is the horizontal sum of the assigned lattices $\mathcal{L}(\mathcal{A}_i)$, $i \in I$.

Theorem 5 *Let a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ be a horizontal sum of basic algebras $\mathcal{A}_i$ $(i \in I)$. Let $\exists_i$ be an existential quantifier on $\mathcal{A}_i$, i.e. every $(A_i; \oplus, \neg, \exists_i, 0)$ is a monadic basic algebra. Let $\exists\colon A \to A$ be a mapping whose restriction on each A_i is equal to $\exists_i$. Then $\mathcal{A}_\exists = (A; \oplus, \neg, \exists, 0)$ is a monadic basic algebra.*

Proof We must check the axioms (e1) – (e4) for $\exists$ on the assigned lattice $\mathcal{L}(A)$. Trivially, we have $\exists 0 = 0$, $\exists(\exists x) = \exists x$ and $x \leq \exists x$. For $x, y \in A_i$ we have $\exists(x \vee y) = \exists x \vee \exists y$ by the definition. If $x \in A_i$, $y \in A_j$ for $i \neq j$ then $x \vee y = 1$ but also $\exists x \vee \exists y = 1$ thus, by (i) of Lemma 1, $1 = \exists(x \vee y) = \exists x \vee \exists y$. To check the condition (e4) is almost trivial since $x \leq y$ only if $x, y \in A_i$ and, inside A_i, it holds by the definition. $\square$

Example 3 Consider the basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, where $A = \{0, a, b, c, 1\}$, whose assigned lattice $\mathcal{L}(A)$ is in Fig. 3.

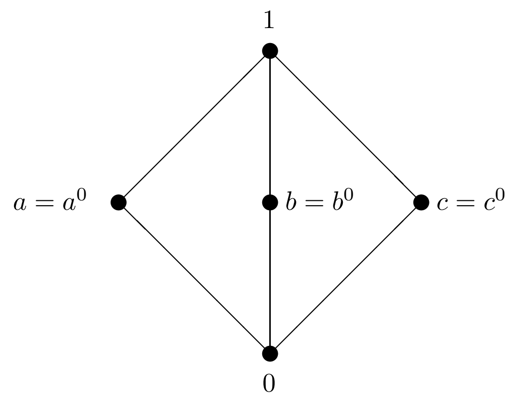

Fig. 3

Clearly, $\mathcal{A}$ is a horizontal sum of $\mathcal{H}$ (see Example 2) and the three element chain MV-algebra $\{0, c, 1\}$. Let us note that also $\mathcal{H}$ is a horizontal sum of two three element chain MV-algebras. One can easily verify that the mapping h defined by

$$h(a) = a, \quad h(b) = b, \quad h(c) = 1, \quad h(0) = 0, \quad h(1) = 1$$

is an e-retraction on $\mathcal{L}(A)$. In fact, h is composed by the e-retraction id on $\mathcal{H}$ and j on $\{0, c, 1\}$. The subalgebra $\mathcal{A}_0 = h(A)$ is clearly $\{0, a, b, 1\}$ (which is isomorphic to $\mathcal{H}$). It is an m-relatively complete subalgebra of $\mathcal{A}$.

Example 4 Consider again the basic algebra from Example 3. Let $B = \{0, c, 1\}$. It is a routine way to check that B is a relatively complete subalgebra of $\mathcal{A}$ and, by Theorem 4, it induces an existential quantifier. Of course, $\exists 0 = 0$, $\exists 1 = 1$ and we can easily compute

$$\exists a = \inf\{x \in B;\ a \leq x\} = 1,$$
$$\exists b = \inf\{x \in B;\ b \leq x\} = 1,$$
$$\exists c = \inf\{x \in B;\ c \leq x\} = c.$$

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