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On Structure Space of Γ-Semigroups

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Abstract

In this paper we introduce the notion of the structure space of Γ-semigroups formed by the class of uniformly strongly prime ideals. We also study separation axioms and compactness property in this structure space.

Key words: Γ-semigroup; uniformly strongly prime ideal; Noetherian Γ-semigroup, hull-kernel topology, structure space.

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1 Introduction

In [4], L. Gillman studied “Rings with Hausdorff structure space” and in [7], C. W. Kohls studied “The space of prime ideals of a ring”. In [1], M. R. Adhikari and M. K. Das studied ‘Structure spaces of semirings’.

In [9], M. K. Sen and N. K. Saha introduced the notion of Γ-Semigroup. Some works on Γ-Semigroups may be found in [10], [8], [5], [6], [2] and [3].

In this paper we introduce and study the structure space of Γ-Semigroups. For this we consider the collection $\mathcal{A}$ of all proper uniformly strongly prime ideals of a Γ-Semigroup S and we give a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ by means of closure operator defined in terms of intersection and inclusion relation among these ideals of the Γ-Semigroup S. We call the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$—the structure space of the Γ-Semigroup S. We study separation axioms, compactness and connectedness in this structure space.

2 Preliminaries

Definition 2.1 Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if
(i) $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
S is said to be Γ-semigroup with zero if there exists an element $0 \in S$ such that $0\alpha a = a\alpha 0 = 0$ for all $\alpha \in \Gamma$.

Example 2.2 Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let

$$\Gamma = \{-\tfrac{1}{p} : p \text{ is prime}\}.$$

Let $a, b, c \in S$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup.

Definition 2.3 Let S be a Γ-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α-idempotent if $e\alpha e = e$. The set of all α-idempotents is denoted by E_α and we denote $\bigcup_{\alpha\in\Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S.

Definition 2.4 A nonempty subset I of a Γ-semigroup S is called an ideal if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$ where for subsets U, V of S and Δ of Γ, $U\Delta V = \{u\alpha v : u \in U, v \in V, \alpha \in \Delta\}$.

Definition 2.5 A nonempty subset I of a Γ-semigroup S is called an ideal if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$ where for subsets U, V of S and Δ of Γ, $U\Delta V = \{u\alpha v : u \in U, v \in V, \alpha \in \Delta\}$. An ideal I of S is called a proper ideal if $I \neq S$.

Definition 2.6 A proper ideal P of a Γ-Semigroup S is called a prime ideal of S if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any two ideals A, B of S.

Definition 2.7 An ideal I of a Γ-semigroup S is said to be full if $E(S) \subseteq I$.

An ideal I of a Γ-semigroup S is said to be a prime full ideal if it is both prime and full.

Theorem 2.8 *Let S be a Γ-semigroup. For an ideal P of S, the following are equivalent.*
(i) *If A and B are ideals of S such that $A\Gamma B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.*
(ii) *If $a\Gamma S\Gamma b \subseteq P$ then either $a \in P$ or $b \in P (a, b \in S)$*
(iii) *If I_1 and I_2 are two right ideals of S such that $I_1\Gamma I_2 \subseteq P$ then either $I_1 \subseteq P$ or $I_2 \subseteq P$.*
(iv) *If J_1 and J_2 are two left ideals of S such that $J_1\Gamma J_2 \subseteq P$ then either $J_1 \subseteq P$ or $J_2 \subseteq P$.*

Proof (i) $\Rightarrow$ (ii): Suppose $a\Gamma S\Gamma b \subseteq P$. Then $<a>\Gamma<a>\Gamma<b>\Gamma<b> \subseteq P$. Since $<a>\Gamma<a>$, $<b>\Gamma<b>$ are ideals of S, so by (i) we have either $<a>\Gamma<a> \subseteq P$ or $<b>\Gamma<b> \subseteq P$. By repeated uses of (i) we get $a \in <a> \subseteq P$ or $b \in <b> \subseteq P$.

(ii) $\Rightarrow$ (iii): Let $I_1\Gamma I_2 \subseteq P$. Let $I_1 \nsubseteq P$. Then there exists an element $a_1 \in I_1$ such that $a_1 \notin P$. Then for every $a_2 \in I_2$ we have $a_1\Gamma S\Gamma a_2 \subseteq I_1\Gamma I_2 \subseteq P$. Hence from (ii) $a_2 \in P$. Thus $I_2 \subseteq P$. Similarly (ii) implies (iv).

The proof is completed by observing that (i) is implied obviously either by (iii) or by (iv). □

Definition 2.9 An ideal P of a Γ-Semigroup S is called a uniformly strongly prime ideal(usp ideal) if S and Γ contain finite subsets F and Δ respectively such that $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in S$.

Theorem 2.10 *Let S be a Γ-semigroup. Then every uniformly strongly prime ideal is a prime ideal.*

Proof Let P be a uniformly strongly prime ideal of S. Then S and Γ contain finite subsets F and Δ respectively such that $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in S$. Now let $a\Gamma S\Gamma b \subseteq P$. Thus we have $a\Delta F\Delta b \subseteq a\Gamma S\Gamma b \subseteq P$ and hence we have $a \in P$ or $b \in P$. Hence P is prime ideal by Theorem 2.8. □

Throughout this paper S will always denote a Γ-Semigroup with zero and unless otherwise stated a Γ-Semigroup means a Γ-Semigroup with zero.

3 Structure space of Γ-semigroups

Suppose $\mathcal{A}$ is the collection of all uniformly strongly prime ideals of a Γ-Semigroup S. For any subset A of $\mathcal{A}$, we define

$$\overline{A} = \{I \in \mathcal{A} \colon \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}.$$

It is easy to see that $\overline{\emptyset} = \emptyset$.

Theorem 3.1 *Let A, B be any two subsets of $\mathcal{A}$. Then*

(i) $A \subseteq \overline{A}$

(ii) $\overline{\overline{A}} = \overline{A}$

(iii) $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$

(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof (i): Clearly, $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ for each α and hence $A \subseteq \overline{A}$.

(ii): By (i), we have $\overline{A} \subseteq \overline{\overline{A}}$. For converse part, let $I_\beta \in \overline{\overline{A}}$. Then $\bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. Now $I_\alpha \in \overline{A}$ implies that $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$ for all $\alpha \in \Lambda$. Thus

$$\bigcap_{I_\gamma \in A} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta \qquad \text{i.e.} \qquad \bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\beta.$$

So $I_\beta \in \overline{A}$ and hence $\overline{\overline{A}} \subseteq \overline{A}$. Consequently, $\overline{\overline{A}} = \overline{A}$.

(iii): Suppose that $A \subseteq B$. Let $I_\alpha \in \overline{A}$. Then $\bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha$. Since $A \subseteq B$, it follows that

$$\bigcap_{I_\beta \in B} I_\beta \subseteq \bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha.$$

This implies that $I_\alpha \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(iv): Clearly, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

For the reverse part, let $I_\alpha \in \overline{A \cup B}$. Then $\bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha$.
It is easy to see that

$$\bigcap_{I_\beta \in A \cup B} I_\beta = \Big(\bigcap_{I_\beta \in A} I_\beta\Big) \cap \Big(\bigcap_{I_\beta \in B} I_\beta\Big).$$

Since $\bigcap_{I_\beta \in A} I_\beta$ and $\bigcap_{I_\beta \in B} I_\beta$ are ideals of S, we have

$$\Big(\bigcap_{I_\beta \in A} I_\beta\Big)\Gamma\Big(\bigcap_{I_\beta \in B} I_\beta\Big) \subseteq \Big(\bigcap_{I_\beta \in A} I_\beta\Big) \cap \Big(\bigcap_{I_\beta \in B} I_\beta\Big) = \bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha$$

Since every uniformly strongly prime ideal is prime, I_α is a prime ideal of S and hence either $\bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha$ or $\bigcap_{I_\beta \in B} I_\beta \subseteq I_\alpha$ i.e. either $I_\alpha \in \overline{A}$ or $I_\alpha \in \overline{B}$ i.e. $I_\alpha \in \overline{A} \cup \overline{B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ and hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

Definition 3.2 The closure operator $A \longrightarrow \overline{A}$ gives a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$. This topology $\tau_{\mathcal{A}}$ is called the hull-kernel topology and the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called the structure space of the Γ-Semigroup S.

Let I be a ideal of a Γ-Semigroup S. We define

$$\Delta(I) = \{I' \in \mathcal{A} \colon I \subseteq I'\} \quad \text{and} \quad C\Delta(I) = \mathcal{A} \setminus \Delta(I) = \{I' \in \mathcal{A} \colon I \not\subseteq I'\}.$$

Now we have the following result:

Proposition 3.3 *Any closed set in $\mathcal{A}$ is of the form $\Delta(I)$, where I is a ideal of a Γ-Semigroup S.*

Proof Let $\overline{A}$ be any closed set in $\mathcal{A}$, where $A \subseteq \mathcal{A}$. Let $A = \{I_\alpha \colon \alpha \in \Lambda\}$ and $I = \bigcap_{I_\alpha \in A} I_\alpha$. Then I is a ideal of S. Let $I' \in \overline{A}$. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I'$. This implies that $I \subseteq I'$. Consequently, $I' \in \Delta(I)$. So $\overline{A} \subseteq \Delta(I)$.

Conversely, let $I' \in \Delta(I)$. Then $I \subseteq I'$ i.e. $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I'$. Consequently, $I' \in \overline{A}$ and hence $\Delta(I) \subseteq \overline{A}$. Thus $\overline{A} = \Delta(I)$. □

Corollary 3.4 *Any open set in $\mathcal{A}$ is of the form $C\Delta(I)$, where I is an ideal of S.*

Let S be a Γ-Semigroup and $a \in S$. We define

$$\Delta(a) = \{I \in \mathcal{A} : a \in I\} \quad \text{and} \quad C\Delta(a) = \mathcal{A} \setminus \Delta(a) = \{I \in \mathcal{A} : a \notin I\}.$$

Then we have the following result:

Proposition 3.5 *$\{C\Delta(a) : a \in S\}$ forms an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.*

Proof Let $U \in \tau_{\mathcal{A}}$. Then $U = C\Delta(I)$, where I is an ideal of S. Let $J \in U = C\Delta(I)$. Then $I \not\subseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Thus $J \in C\Delta(a)$. Now it remains to show that $C\Delta(a) \subset U$. Let $K \in C\Delta(a)$. Then $a \notin K$. This implies that $I \not\subseteq K$. Consequently, $K \in U$ and hence $C\Delta(a) \subset U$. So we find that $J \in C\Delta(a) \subset U$. Thus $\{C\Delta(a) : a \in S\}$ is an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.

Theorem 3.6 *The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0-space.*

Proof Let I_1 and I_2 be two distinct elements of $\mathcal{A}$. Then there is an element a either in $I_1 \setminus I_2$ or in $I_2 \setminus I_1$. Suppose that $a \in I_1 \setminus I_2$. Then $C\Delta(a)$ is a neighbourhood of I_2 not containing I_1. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0-space. □

Theorem 3.7 *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1-space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.*

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a T_1-space. Suppose that I_1 and I_2 be any two distinct elements of $\mathcal{A}$. Then each of I_1 and I_2 has a neighbourhood not containing the other. Since I_1 and I_2 are arbitrary elements of $\mathcal{A}$, it follows that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

Conversely, suppose that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$. Let I_1 and I_2 be any two distinct elements of $\mathcal{A}$. Then by hypothesis, $I_1 \not\subset I_2$ and $I_2 \not\subset I_1$. This implies that there exist $a, b \in S$ such that $a \in I_1$ but $a \notin I_2$ and $b \in I_2$ but $b \notin I_1$. Consequently, we have $I_1 \in C\Delta(b)$ but $I_1 \notin C\Delta(a)$ and $I_2 \in C\Delta(a)$ but $I_2 \notin C\Delta(b)$ i.e. each of I_1 and I_2 has a neighbourhood not containing the other. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1-space. □

Corollary 3.8 *Let $\mathcal{M}$ be the set of all proper maximal ideals of a Γ-Semigroup S with unities. Then $(\mathcal{M}, \tau_{\mathcal{M}})$ is a T_1-space, where $\tau_{\mathcal{M}}$ is the induced topology on $\mathcal{M}$ from $(\mathcal{A}, \tau_{\mathcal{A}})$.*

Theorem 3.9 *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space if and only if for any two distinct pair of elements I, J of $\mathcal{A}$, there exist $a, b \in S$ such that $a \notin I$, $b \notin J$ and there does not exist any element K of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$.*

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a Hausdorff space. Then for any two distinct elements I, J of $\mathcal{A}$, there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. Now $I \in C\Delta(a)$ and $J \in C\Delta(b)$ imply that $a \notin I$ and $b \notin J$. If possible, let K be any element of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$. Then $K \in C\Delta(a)$, $K \in C\Delta(b)$ and hence $K \in C\Delta(a) \cap C\Delta(b)$, a contradiction, since $C\Delta(a) \cap C\Delta(b) = \emptyset$. Thus there does not exist any element K of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$.

Conversely, suppose that the given condition holds and $I, J \in \mathcal{A}$ such that $I \neq J$. Let $a, b \in S$ be such that $a \notin I$, $b \notin J$ and there does not exist any K of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$. Then $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. This implies that $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space. □

Corollary 3.10 *If $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space, then no proper uniformly strongly prime ideal contains any other proper uniformly strongly prime ideal. If $(\mathcal{A}, \tau_{\mathcal{A}})$ contains more than one element, then there exist $a, b \in S$ such that $\mathcal{A} = C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$, where I is the ideal generated by a, b.*

Proof Suppose that $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space. Since every Hausdorff space is a T_1-space, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1-space. Hence by Theorem 3.7, it follows that no proper uniformly strongly prime ideal contains any other proper uniformly strongly prime ideal. Now let $J, K \in \mathcal{A}$ be such that $J \neq K$. Since $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space, there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $J \in C\Delta(a)$, $K \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. Let I be the ideal generated by a, b. Then I is the smallest ideal containing a and b. Let $K \in \mathcal{A}$. Then either $a \in K$, $b \notin K$ or $a \notin K$, $b \in K$ or $a, b \in K$. The case $a \notin K$, $b \notin K$ is not possible, since $C\Delta(a) \cap C\Delta(b) = \emptyset$. Now in the first case, $K \in C\Delta(b)$ and hence $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. In the second case, $K \in C\Delta(a)$ and hence $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. In the third case, $K \in \Delta(I)$ and hence $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. So we find that $\mathcal{A} \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. Again, clearly $C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \subseteq \mathcal{A}$. Hence $\mathcal{A} = C\Delta(a) \cup C\Delta(b) \cup \Delta(I)$. □

Theorem 3.11 *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space if and only if for any $I \in \mathcal{A}$ and $a \notin I$, $a \in S$, there exist an ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.*

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a regular space. Let $I \in \mathcal{A}$ and $a \notin I$. Then $I \in C\Delta(a)$ and $\mathcal{A} \setminus C\Delta(a)$ is a closed set not containing I. Since $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space, there exist disjoints open sets U and V such that $I \in U$ and $\mathcal{A} \setminus C\Delta(a) \subseteq V$. This implies that $\mathcal{A} \setminus V \subseteq C\Delta(a)$. Since V is open, $\mathcal{A} \setminus V$ is closed and hence there exists an ideal J of S such that $\mathcal{A} \setminus V = \Delta(J)$, by Proposition 3.3. So we find that $\Delta(J) \subseteq C\Delta(a)$. Again, since $U \cap V = \emptyset$, we have $V \subseteq \mathcal{A} \setminus U$. Since U is open, $\mathcal{A} \setminus U$ is closed and hence there exists an ideal K of S such that $\mathcal{A} \setminus U = \Delta(K)$ i.e. $V \subseteq \Delta(K)$. Since $I \in U$, $I \notin \mathcal{A} \setminus U = \Delta(K)$. This implies that $K \nsubseteq I$. Thus there exists $b \in K (\subset S)$ such that $b \notin I$. So $I \in C\Delta(b)$. Now we show that $V \subseteq \Delta(b)$. Let $M \in V \subseteq \Delta(K)$. Then $K \subseteq M$. Since $b \in K$, it follows that $b \in M$ and hence $M \in \Delta(b)$. Consequently, $V \subseteq \Delta(b)$. This implies that $\mathcal{A} \setminus \Delta(b) \subseteq \mathcal{A} \setminus V = \Delta(J) \Longrightarrow C\Delta(b) \subseteq \Delta(J)$. Thus we find that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$.

Conversely, suppose that the given condition holds. Let $I \in \mathcal{A}$ and $\Delta(K)$ be any closed set not containing I. Since $I \notin \Delta(K)$, we have $K \not\subset I$. This implies that there exists $a \in K$ such that $a \notin I$. Now by the given condition, there exists an ideal J of S and $b \in S$ such that $I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a)$. Since $a \in K$, $C\Delta(a) \cap \Delta(K) = \emptyset$. This implies that $\Delta(K) \subseteq \mathcal{A} \setminus C\Delta(a) \subseteq \mathcal{A} \setminus \Delta(J)$. Since $\Delta(J)$ is a closed set, $\mathcal{A} \setminus \Delta(J)$ is an open set containing the closed set $\Delta(K)$. Clearly, $C\Delta(b) \cap (\mathcal{A} \setminus \Delta(J)) = \emptyset$. So we find that $C\Delta(b)$ and $\mathcal{A} \setminus \Delta(J)$ are two disjoints open sets containing I and $\Delta(K)$ respectively. Consequently, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a regular space. □

Theorem 3.12 *$(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space if and only if for any collection $\{a_\alpha\}_{\alpha \in \Lambda} \subset S$ there exists a finite subcollection $\{a_i : i = 1, 2, \ldots, n\}$ in S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.*

Proof Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a compact space. Then the open cover $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ of $(\mathcal{A}, \tau_{\mathcal{A}})$ has a finite subcover $\{C\Delta(a_i) : i = 1, 2, \ldots, n\}$. Let I be any element of $\mathcal{A}$. Then $I \in C\Delta(a_i)$ for some $a_i \in S$. This implies that $a_i \notin I$. Hence $\{a_i : i = 1, 2, \ldots, n\}$ is the required finite subcollection of elements of S such that for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$.

Conversely, suppose that the given condition holds. Let $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ be an open cover of $\mathcal{A}$. Suppose to the contrary that no finite subcollection of $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ covers $\mathcal{A}$. This means that for any finite set $\{a_1, a_2, \ldots, a_n\}$ of elements of S,

$$
\begin{aligned}
& C\Delta(a_1) \cup C\Delta(a_2) \cup \ldots \cup C\Delta(a_n) \neq \mathcal{A} \\
\Longrightarrow \quad & \Delta(a_1) \cap \Delta(a_2) \cap \ldots \cap \Delta(a_n) \neq \emptyset \\
\Longrightarrow \quad & \text{there exists } I \in \mathcal{A} \text{ such that } I \in \Delta(a_1) \cap \Delta(a_2) \cap \ldots \cap \Delta(a_n) \\
\Longrightarrow \quad & a_1, a_2, \ldots, a_n \in I, \text{ which contradicts our hypothesis .}
\end{aligned}
$$

So the open cover $\{C\Delta(a_\alpha) : a_\alpha \in S\}$ has a finite subcover and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.

Corollary 3.13 *If S is finitely generated, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space.*

Proof Let $\{a_i : i = 1, 2, \ldots, n\}$ be a finite set of generators of S. Then for any $I \in \mathcal{A}$, there exists a_i such that $a_i \notin I$, since I is a proper uniformly strongly prime ideal of S. Hence by Theorem 3.12, $(\mathcal{A}, \tau_{\mathcal{A}})$ is a compact space. □

Definition 3.14 A Γ-Semigroup S is called a Noetherian Γ-Semigroup if it satisfies the ascending chain condition on ideals of S i.e. if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ is an ascending chain of ideals of S, then there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$.

Theorem 3.15 *If S is a Noetherian Γ-Semigroup, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact.*

Proof Let $\{\Delta(I_n)\}_{n=1}^{\infty}$ be a countable collection of closed sets in $\mathcal{A}$ with finite intersection property (FIP). Let us consider the following ascending chain of prime ideals of S: $<I_1> \subseteq <I_1 \cup I_2> \subseteq <I_1 \cup I_2 \cup I_3> \subseteq \ldots$

Since S is a Noetherian Γ-Semigroup, there exists a positive integer m such that $<I_1 \cup I_2 \cup \ldots \cup I_m> = <I_1 \cup I_2 \cup \ldots \cup I_{m+1}> = \ldots$

Thus it follows that $<I_1 \cup I_2 \cup \ldots \cup I_m> \in \bigcap_{n=1}^{\infty} \Delta(I_n)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(I_n) \neq \emptyset$ and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is countably compact. $\square$

Corollary 3.16 *If S is a Noetherian Γ-Semigroup and $(\mathcal{A}, \tau_{\mathcal{A}})$ is second countable, then $(\mathcal{A}, \tau_{\mathcal{A}})$ is compact.*

Proof Proof follows from Theorem 3.15 and the fact that a second countable space is compact if it is countably compact. $\square$

Remark 3.17 Let $\{I_\alpha\}$ be a collection of prime ideals of a Γ-semigroup S. Then $\bigcap I_\alpha$ is an ideal of S but it may not be a prime ideal of S, in general.

However; in particular, we have the following result:

Proposition 3.18 *Let $\{I_\alpha\}$ be a collection of prime ideals of a Γ-semigroup S such that $\{I_\alpha\}$ forms a chain. Then $\bigcap I_\alpha$ is a prime ideal of S.*

Proof Clearly, $\bigcap I_\alpha$ is an ideal of S. Let $A\Gamma B \subseteq \bigcap I_\alpha$ for any two ideals A, B of S. If possible, let $A, B \not\subseteq \bigcap I_\alpha$. Then there exist α and β such that $A \not\subseteq I_\alpha$ and $B \not\subseteq I_\beta$. Since I_α is a chain, let $I_\alpha \subseteq I_\beta$. This implies that $B \not\subseteq I_\alpha$. Since $A\Gamma B \subseteq I_\alpha$ and I_α is prime, we must have either $A \subseteq I_\alpha$ or $B \subseteq I_\alpha$, a contradiction. Therefore, either $A \subseteq \bigcap I_\alpha$ or $B \subseteq \bigcap I_\alpha$. Consequently, $\bigcap I_\alpha$ is a prime ideal of S. $\square$

Definition 3.19 The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called irreducible if for any decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed subsets of $\mathcal{A}$, we have either $\mathcal{A} = \mathcal{A}_1$ or $\mathcal{A} = \mathcal{A}_2$.

Theorem 3.20 *Let A be a closed subset of $\mathcal{A}$. Then A is irreducible if and only if $\bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of S.*

Proof Let A be irreducible. Let P and Q be two ideals of S such that $P\Gamma Q \subseteq \bigcap_{I_\alpha \in A} I_\alpha$. Then $P\Gamma Q \subseteq I_\alpha$ for all α. Since I_α is prime, either $P \subseteq I_\alpha$ or $Q \subseteq I_\alpha$ which implies for $I_\alpha \in A$ either $I_\alpha \in \{\overline{P}\}$ or $I_\alpha \in \{\overline{Q}\}$. Hence $A = (A \cap \overline{P}) \cup (A \cap \overline{Q})$. Since A is irreducible and $(A \cap \overline{P})$, $(A \cap \overline{Q})$ are closed, it follows that $A = A \cap \overline{P}$ or $A = A \cap \overline{Q}$ and hence $A \subseteq \overline{P}$ or $A \subseteq \overline{Q}$. This implies that $P \subseteq \bigcap_{I_\alpha \in A} I_\alpha$ or $Q \subseteq \bigcap_{I_\alpha \in A} I_\alpha$. Consequently, $\bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of S.

Conversely, suppose that $\bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of S. Let $A = A_1 \cup A_2$, where A_1 and A_2 are closed subsets of A. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq \bigcap_{I_\alpha \in A_1} I_\alpha$ and $\bigcap_{I_\alpha \in A} I_\alpha \subseteq \bigcap_{I_\alpha \in A_2} I_\alpha$. Also

$$\bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_1 \cup A_2} I_\alpha = \Big(\bigcap_{I_\alpha \in A_1} I_\alpha \Big) \cap \Big(\bigcap_{I_\alpha \in A_2} I_\alpha \Big).$$

Now

$$\Big(\bigcap_{I_\alpha\in A_1} I_\alpha\Big)\Gamma\Big(\bigcap_{I_\alpha\in A_2} I_\alpha\Big)\subseteq\Big(\bigcap_{I_\alpha\in A_1} I_\alpha\Big)\quad\text{and}\quad\Big(\bigcap_{I_\alpha\in A_1} I_\alpha\Big)\Gamma\Big(\bigcap_{I_\alpha\in A_2} I_\alpha\Big)\subseteq\Big(\bigcap_{I_\alpha\in A_2} I_\alpha\Big).$$

Thus we have

$$\Big(\bigcap_{I_\alpha\in A_1} I_\alpha\Big)\Gamma\Big(\bigcap_{I_\alpha\in A_2} I_\alpha\Big)\subseteq\Big(\bigcap_{I_\alpha\in A_1} I_\alpha\Big)\cap\Big(\bigcap_{I_\alpha\in A_2} I_\alpha\Big).$$

Since $\bigcap_{I_\alpha\in A} I_\alpha$ is prime, it follows that either

$$\bigcap_{I_\alpha\in A_1} I_\alpha\subseteq\bigcap_{I_\alpha\in A} I_\alpha\quad\text{or}\quad\bigcap_{I_\alpha\in A_2} I_\alpha\subseteq\bigcap_{I_\alpha\in A} I_\alpha.$$

So we find that

$$\bigcap_{I_\alpha\in A} I_\alpha=\bigcap_{I_\alpha\in A_1} I_\alpha\quad\text{or}\quad\bigcap_{I_\alpha\in A} I_\alpha=\bigcap_{I_\alpha\in A_2} I_\alpha.$$

Let $I_\beta\in A$. Then we have

$$\bigcap_{I_\alpha\in A_1} I_\alpha\subseteq I_\beta\quad\text{or}\quad\bigcap_{I_\alpha\in A_2} I_\alpha\subseteq I_\alpha.$$

Since $A_1, A_2\subseteq A$, so either $I_\alpha\subseteq I_\beta$ for all $I_\alpha\in A_1$ or $I_\alpha\subseteq I_\beta$ for all $I_\alpha\in A_2$. Thus $I_\beta\in\overline{A_1}=A_1$ or $I_\beta\in\overline{A_2}=A_2$, since A_1 and A_2 are closed. i.e. $A=A_1$ or $A=A_2$. □

Let $\mathcal{C}$ be the collection of all uniformly strongly prime full ideals of a Γ-semigroup S. Then we see that $\mathcal{C}$ is a subset of $\mathcal{A}$ and hence $(\mathcal{C},\tau_{\mathcal{C}})$ is a topological space, where $\tau_{\mathcal{C}}$ is the subspace topology.

In general, $(\mathcal{A},\tau_{\mathcal{A}})$ is not compact and connected. But in particular, for the topological space $(\mathcal{C},\tau_{\mathcal{C}})$, we have the following results:

Theorem 3.21 *$(\mathcal{C},\tau_{\mathcal{C}})$ is a compact space.*

Proof Let $\{\Delta(I_\alpha)\colon\alpha\in\Lambda\}$ be any collection of closed sets in $\mathcal{C}$ with finite intersection property. Let I be the uniformly strongly prime full ideal generated by $E(S)$. Since any uniformly strongly prime full ideal J contains $E(S)$, J contains I. Hence $I\in\bigcap_{\alpha\in\Lambda}\Delta(I_\alpha)\neq\emptyset$. Consequently, $(\mathcal{C},\tau_{\mathcal{C}})$ is a compact space. □

Theorem 3.22 *$(\mathcal{C},\tau_{\mathcal{C}})$ is a connected space.*

Proof Let I be the uniformly strongly prime ideal generated by $E(S)$. Since any uniformly strongly prime full ideal J contains $E(S)$, J contains I. Hence I belongs to any closed set $\Delta(I')$ of $\mathcal{C}$. Consequently, any two closed sets of $\mathcal{C}$ are not disjoint. Hence $(\mathcal{C},\tau_{\mathcal{C}})$ is a connected space. □

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