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A Visual Approach to Test Lattices*

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Abstract

Let p be a k-ary lattice term. A k-pointed lattice $L = (L; \vee, \wedge, d_1, \dots, d_k)$ will be called a *p-lattice* (or a *test lattice* if p is not specified), if $(L; \vee, \wedge)$ is generated by $\{d_1, \dots, d_k\}$ and, in addition, for any k-ary lattice term q satisfying $p(d_1, \dots, d_k) \le q(d_1, \dots, d_k)$ in L, the lattice identity $p \le q$ holds in all lattices.

In an elementary visual way, we construct a finite p-lattice $L(p)$ for each p. If p is a canonical lattice term, then $L(p)$ coincides with the optimal p-lattice of Freese, Ježek and Nation [6]. Some results on test lattices and short proofs for known facts on free lattices indicate that our approach is useful.

Key words: Free lattice, test lattice, lattice identity, Whitman's condition.

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1 Introduction

For a fixed natural number k, by a *k-pointed lattice* we mean a lattice L with k distinguished elements $d_1, \dots, d_k$. For $\vec{d} = (d_1, \dots, d_k) \in L^k$, the "$k$-pointed lattice" $(L; \vee, \wedge, d_1, \dots, d_k)$ will be denoted by $(L; \vec{d})$. If p and q are k-ary lattice terms, then both $p = q$ and $p \le q$ are called *lattice identities.* A lattice identity is said to be *trivial*, if it holds in all lattices.

We introduce a new concept. Given a k-ary lattice term $p = p(\alpha_1, \dots, \alpha_k)$, we will call a k-pointed lattice $(L; \vec{d})$ a *p-lattice*, if

- $\{d_1, \dots, d_k\}$ generates L, and
- for any k-ary lattice term q, $p(d_1, \dots, d_k) \le q(d_1, \dots, d_k)$ in L if and only if $p \le q$ is a trivial lattice identity.

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We use the terminology *"test lattice"* if we do not want to specify p. That is, if $(L;\vec{d})$ is a p-lattice for some p, then it is also called a test lattice.

For example, if L is freely generated by $\{d_1,\dots,d_k\}$, then it is obviously a p-lattice for every k-ary lattice term p. Beside other aims, we are going to give a new proof for the following result, which is not so obvious.

Proposition 1 [Freese and Nation [7], Freese, Ježek and Nation [6])] *For each lattice term p, there exists a* finite *p-lattice.*

Our first goal is to point out that test lattices *deserve some attention* independently from the well-developed theory of free lattices (see Freese, Ježek, Nation [6]). Hence we present Theorem 3, soon, and give new proofs for two more or less known properties of test lattices, see Theorems 5 and 6. Further, we give two easy applications. Namely, we demonstrate the usefulness of test lattices by giving a very short, new proof that free lattices satisfy Whitman's condition, see Corollaries 12 and 13, and also by solving (and generalizing) the following (not very difficult) exercise.

Exercise 2 *Let $p^{\diamondsuit}=(\alpha_1\vee\alpha_2)\wedge(\alpha_1\vee\alpha_3)$. Is there a non-trivial lattice identity $p^{\diamondsuit}\leq q$ that holds in the five-element non-modular lattice?*

Our second goal is to *construct* a finite k-pointed lattice $L(p)$, for each k-ary lattice term p, in a *conceptually simple way*, and to give an *elementary proof* that it is a p-lattice. To follow the rest of the paper until the "Historical remarks" section, the reader is assumed to be familiar only with the rudiments of lattice theory, that is, with a small fraction of, say, G. Grätzer [8]. The only outer reference used in our proof is Jónsson's type 3 representation theorem, see [10], and see also Theorem IV.4.4 in Grätzer [8].

Our third goal is to give a new approach that is *visual*, not just elementary. We *develop a visual toolkit* consisting of purely lattice theoretical results from this section and several statements (Lemmas 7, 8, 9, 14, 15 and Corollaries 10, 11) from Section 3. Although this toolkit is applied to prove some known or easy results only, the geometric perspective may serve a better understanding of the underlining reasons, and it may lead to further useful observations in the future.

Notice at this point that powerful tools from the theory of free lattices, see Freese, Ježek and Nation [6] and its references, have already given or may easily give shorter "standard" proofs to several of our statements. Hence, in the last section, our results will be related to [6]. However, if the necessary previous pages of [6] are also counted, then some of the standard proofs are lengthier than ours. Although we will give some hints to a standard proof in the last section, many readers will probably find easier to follow our approach.

Notice also that, opposed to the present paper, free lattices are hard to imagine visually. For example, $FL(\omega)$ is a sublattice of $FL(3)$ by Whitman [13], and this fact is an obstacle to a proper visual understanding of $FL(3)$, the free lattice on three generators. Hence we hope that our pictorial approach with graphical background makes sense and contributes to a better understanding of free lattices.

Finally, notice at this point that the only outer reference, Jónsson's type 3 representation theorem, see [10] or Theorem IV.4.4 in [8], is also visual.

From now on, let $p = p(\alpha_1, \ldots, \alpha_k)$ be a fixed k-ary lattice term. We are going to construct a k-pointed lattice $L(p) = (L(p); d_1, \ldots, d_k)$ such that the following theorem holds.

Theorem 3 *$L(p) = (L(p); d_1, \ldots, d_k)$ is a finite p-lattice.*

By an *optimal p-lattice*, we mean a p-lattice that is a k-pointed lattice homomorphic image of any other p-lattice. The following corollary of Theorem 3 is straightforward and more or less evident.

Corollary 4 *For each lattice term p, there exists an optimal p-lattice $K(p)$. It is finite and it is unique up to k-pointed lattice isomorphism.*

The length of a lattice term q, to be defined in the usual syntactical way later, will be denoted by length(q). We say that p is a *canonical lattice term* if for every k-ary lattice terms q, $p =_{\mathrm{triv}} q$ implies length$(p) \leq$ length(q). Like every term, each canonical lattice term p is

- either a variable,
- or the meet of at least two terms,
- or the join of at least two terms.

In the first two cases we say that p is a *join-irreducible canonical term.* (This means that p represent a join-irreducible element of the free lattice generated by $\{\alpha_1, \ldots, \alpha_k\}$.)

Unfortunately, $L(p)$ is usually not an *optimal* p-lattice in general. For example, for $p^\natural = \big(\big((\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_2 \vee \alpha_3)\big) \vee \alpha_2\big) \wedge \alpha_4$, the $p^\natural$-lattice $L(p^\natural)$ is not optimal. As a compensation, we have the following two theorems.

Theorem 5 [essentially in Freese, Ježek and Nation [6]] *If p is a canonical lattice term, then $L(p)$ equals $K(p)$, the optimal p-lattice.*

Theorem 6 [Freese, Ježek and Nation [6]] *If p is a join-irreducible canonical lattice term, then $K(p) = L(p)$ is subdirectly irreducible.*

Notice that the assumption of join-irreducibility in Theorem 6 cannot be avoided. For example, $L(\alpha_1 \vee \alpha_2) = K(\alpha_1 \vee \alpha_2)$ is the four-element boolean lattice, which is subdirectly (and even directly) reducible. On the other hand, this assumption is not so restrictive. Indeed, if p is the join of its subterms $p_1, \ldots, p_n$, then, evidently, $p \leq_{\mathrm{triv}} q$ iff $p_i \leq_{\mathrm{triv}} q$ for $i = 1, \ldots, n$. Hence, to investigate if $p \leq_{\mathrm{triv}} q$, we can use the subdirectly irreducible $L(p_1)$, ..., $L(p_n)$ instead of $L(p)$.

2 The construction of $L(p)$

We fix a set $X = \{\alpha_1, \dots, \alpha_k\}$ of variables. Since we do not want to make a distinction between lattice terms that differ only modulo commutativity, associativity and idempotency, we give the following inductive definition of $T(X)$, the *set of lattice terms over* X.

- Every $\alpha_i \in X$ is a *doubly irreducible* member of $T(X)$ with $\text{length}(\alpha_i) = 1$.
- Each element of $T(X) \setminus X$ is of length > 1, and it is either join-irreducible and meet-reducible, or meet-irreducible and join-reducible.
- If $q_1, \dots, q_n$, $n \geq 2$, are *distinct* meet-irreducible members of $T(X)$ then $q = \bigwedge_{i=1}^{n} q_i$ belongs to $T(X)$. It is join-irreducible and meet-reducible, and we have $\text{length}(q) = 1 + \sum_{i=1}^{n} \text{length}(q_i)$. The terms $q_1, \dots, q_n$ are called the *meetands* of p.
- If $q_1, \dots, q_n$, $n \geq 2$, are *distinct* join-irreducible members of $T(X)$ then $q = \bigvee_{i=1}^{n} q_i$ belongs to $T(X)$. It is meet-irreducible and join-reducible, and we have $\text{length}(q) = 1 + \sum_{i=1}^{n} \text{length}(q_i)$. The terms $q_1, \dots, q_n$ are called the *joinands* of p.
- Each member of $T(X)$ is obtained by the previous rules in a finite number of steps.

Notice that for each $q \in T(X)$, either q has no meetand or it has at least two meetands. Dually, the same holds for the joinands of q. For concrete terms in examples, we will write $q_1 \vee \cdots \vee q_n$ rather than $\bigvee_{i=1}^{n} q_i$, and similarly for the meet. By a *join-free* term we mean a variable or a meet of variables.

Our definition of terms is only slightly different from that in page 10 of Freese, Ježek and Nation [6]. Namely, $x \vee y \vee z$ and $x \vee (y \vee z)$ are different terms in [6] but $x \vee (y \vee z)$ is *not* a term in the present paper. Notice also that the (ir)reducibility of a term has not much to do with the (ir)reducibility of the corresponding element of the free lattice $FL(X)$. For example, $(\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_2 \vee \alpha_3)$ is a join-irreducible and meet-reducible term, but it represents a join-reducible and meet-irreducible element of $FL(X)$.

The *color set* $C(p)$ of p is defined by the following induction. (The terminology “color” will be clear soon.)

- $C(\alpha_i) = \{\alpha_i\}$
- If p is join-reducible with joinands $p_1, \dots, p_n$, then $C(p) = C(p_1) \cup \cdots \cup C(p_n)$.
- If p is meet-reducible, then let

$$\begin{aligned} M(p) &= \{s : s \text{ is a meetand of } p \text{ with } \text{length}(s) > 1\} \\ &= \{s : s \text{ is a meetand of } p \text{ and } s \text{ is join-reducible}\}, \end{aligned} \tag{1}$$

  and define

$$C(p) = \{p\} \cup \bigcup_{s \in M(p)} C(s)\,.$$

Notice that all elements of $C(p)$ are join-irreducible terms. For an example of $C(p)$, see the set of colors of $H(p^\sharp)$ in Figure 3.

Given a relation E, let E^* denote its transitive closure. Throughout the paper, by a *p-graph* or, shortly, *graph* we mean a structure $G = (V, E, \mathrm{col})$ such that

- $(V, E) = (V(G), E(G))$ is a directed graph without loops and multiple edges. That is, V is a nonempty set, the vertex set, and $E \subseteq V^2$, the edge set, is an irreflexive and antisymmetric relation;
- $\mathrm{col}\colon E \to C(p)$, that is, each edge $e \in V$ has a unique color $\mathrm{col}(e) \in C(p)$;
- E^*, also denoted by $\sqsubset$, is a partial ordering of V with least element, called *the left endpoint of* G, and greatest element, called *the right endpoint*.

Unless otherwise specified, the left and right endpoints of our graphs are denoted by x_0 and x_1, respectively. The subgraphs we are going to consider are also graphs in the above sense. However, a proper subgraph of a p-graph G is (isomorphic with) a q-graph for some term q distinct from p.

In figures, the edges are directed from left to right by convention, so the orientation of edges is not indicated. An edge $(a, b) \in E$ is called a *covering edge* of G, if there is no $c \in V$ with $a \sqsubset c \sqsubset b$. To ease our notations, we will say that (a, r, b) is an "edge of G" to express that $(a, b) \in E$ and $r = \mathrm{col}((a, b))$.

If $\{G_1, G_2\}$ is a two-element set of graphs, then a *4-series connection* of this set is obtained from two copies of G_1 and two copies of G_2, all the four copies being pairwise disjoint, via identifying some endpoints as depicted in Figure 1. Of course, this depends on the order of G_1 and G_2, whence $\{G_1, G_2\}$ has two 4-series connections.

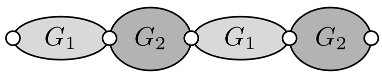

Figure 1: A 4-series connection of $\{G_1, G_2\}$

If $\{G_1, \dots, G_n\}$ is an n-element set of graphs, then each 4-series connection of this set is obtained in the following way: for some $i \in \{1, \dots, n\}$ and a 4-series connection H of $\{G_1, \dots, G_{i-1}, G_{i+1}, \dots, G_n\}$, we form a 4-series connection of H and G_i. Notice that $\{G_1, \dots, G_n\}$ has exactly $n!$ many 4-series connections; for $n = 3$ one of them is depicted in Figure 2

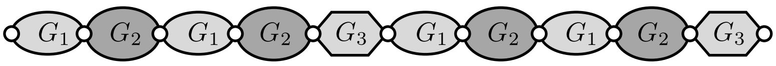

Figure 2: A 4-series connection of $\{G_1, G_2, G_3\}$

Next, we define a sequence $\mathbf{G}_j(p)$ of sets of p-graphs associated with p via induction on j as follows. A particular case,

$$p^\sharp = \alpha_1 \wedge \Big(\alpha_2 \vee \big(\alpha_3 \wedge (\alpha_4 \vee \alpha_5)\big)\Big) \wedge \big(\alpha_2 \vee (\alpha_3 \wedge \alpha_5)\big)$$

is depicted in Figure 3. The reader is advised to look at this figure often while reading the following definition. In Figure 3, $H_j(p^\sharp)$ is just one member of $\mathbf{G}_j(p^\sharp)$.

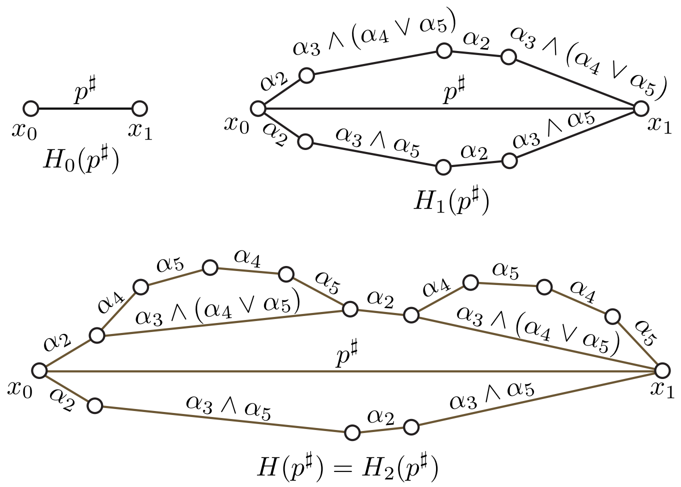

Figure 3: Constructing a member of $\mathbf{G}(p^\sharp)$

If p is join-irreducible, then $\mathbf{G}_0(p)$ consists of a single graph $H_0(p)$. This graph has only two vertices, x_0 and x_1, and only one edge, (x_0, x_1). This edge is colored by p.

If $s = \bigvee_{i=1}^{n} t_i$ is a join-reducible lattice term with joinands $t_1, \ldots, t_n$, then any 4-series connection of the set $\{H_0(t_1), \ldots, H_0(t_n)\}$ is called an s-*arc*; for $n = 3$ see Figure 4.

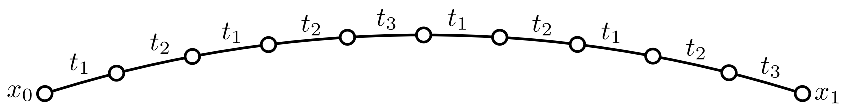

Figure 4: An s-arc, if $s = \bigvee_{i=1}^{3} t_i$

If $p = \bigvee_{i=1}^{n} p_i$ is join-reducible, then let $\mathbf{G}_0(p)$ be the set of all p-arcs.

If $j \geq 1$ and each *covering* edge of every member of $\mathbf{G}_{j-1}(p)$ is colored by a join-free term (variable or meet of variables), then let $\mathbf{G}_j(p) = \mathbf{G}_{j-1}(p)$.

In the opposite case we obtain $\mathbf{G}_j(p)$ from $\mathbf{G}_{j-1}(p)$ in the following way. Take a member $H = H_{j-1}(p) \in \mathbf{G}_{j-1}(p)$. Consider each covering edge (a, r, b) of H whose color r is not join-free. Then r is meet-reducible. For each $s \in M(r)$, see formula (1), we glue an s-arc to H by identifying the left and right endpoints of this arc with a and b, respectively, but keeping other vertices of this arc disjoint from the vertices of H and that of any other arc glued to H. We glue all the necessary arcs to all covering edges with not join-free colors at the same time such that these arcs should be disjoint from each other and from H as much as possible and, in addition,

$$\text{we must use isomorphic } s\text{-arcs for all } r\text{-colored covering edges.} \tag{2}$$

This way we obtain H^+. Finally, let $\mathbf{G}_j(p) = \{H^+ : H \in \mathbf{G}_{j-1}(p)\}$.

If $\mathbf{G}_j(p)$ is different from $\mathbf{G}_{j-1}(p)$, then the maximal length of not join-free colors on covering edges in members of $\mathbf{G}_{j-1}(p)$ decreases when we pass from $\mathbf{G}_{j-1}(p)$ to a $\mathbf{G}_j(p)$. Hence there is a smallest $n \in \mathbf{N}$ with $\mathbf{G}_n(p) = \mathbf{G}_{n-1}(p)$. Let $\mathbf{G}(p) = \mathbf{G}_{n-1}(p)$ for this n. Clearly, the colors of *covering* edges of any member of $\mathbf{G}(p)$ are join-free.

Let us agree on the following convention: $H(p)$ will always denote an arbitrarily *fixed* graph in $\mathbf{G}(p)$. Then $H_j(p)$ will stand for the unique graph in $\mathbf{G}_j(p)$ that occurs in the inductive definition leading to $H(p)$. For technical reasons, $H_{-1}(p)$ will denote the empty graph with no edge.

It is evident from the construction that the set of colors occurring on edges of each $H(p) \in \mathbf{G}(p)$ is exactly $C(p)$.

An edge (a, r, b) of a p-graph $H(p) \in \mathbf{G}(p)$ is called an *α_i-edge* if $r = \alpha_i$ or α_i is a meetand of r. (Notice that an α_i-edge is not necessarily α_i-colored!) Let $V(p)$ and $E(p)$ denote the vertex set and the edge set of $H(p)$, respectively, and let $\mathrm{Equ}(V(p))$ stand for the lattice of equivalences on $V(p)$. The smallest member of $\mathrm{Equ}(V(p))$ collapsing the endpoints of each α_i-edge will be denoted by $\alpha_i|_{H(p)}$. In other words, for $a, b \in V(p)$ we have $(a, b) \in \alpha_i|_{H(p)}$ iff there are vertices $c_0 = a, c_1, \dots, c_n = b$, $n \geq 0$, such that for all $i = 0, 1, \dots, n-1$ either (c_i, c_{i+1}) or (c_{i+1}, c_i) is an α_i-edge. Still in other words: if there is an *undirected path* from a to b whose edges are α_i-edges. Such a path will be called an *α_i-path*.

Finally, the p-lattice we wanted to construct is

$$L(p) = (L(p); d_1, \dots, d_k) := \big([\alpha_1|_{H(p)}, \dots, \alpha_k|_{H(p)}]; \alpha_1|_{H(p)}, \dots, \alpha_k|_{H(p)}\big) \tag{3}$$

where $H(p) \in \mathbf{G}(p)$ and $[\alpha_1|_{H(p)}, \dots, \alpha_k|_{H(p)}]$ is the sublattice of $\mathrm{Equ}(V(p))$ generated by $\{\alpha_1|_{H(p)}, \dots, \alpha_k|_{H(p)}\}$. Since $L(p)$ will be appropriate for any choice of $H(p)$ in $\mathbf{G}(p)$, we will not investigate if $L(p)$ depends on $H(p)$ in the abstract sense or not.

3 Visual statements and proofs

In forthcoming computations, $\leq^{(n)}$, $\leq^{(\mathrm{T}n)}$, $\leq^{(\mathrm{C}n)}$ and $\leq^{(\mathrm{L}n)}$ will indicate that Formula (n), Theorem n, Corollary n and Lemma n is applied, respectively. Analogous superscript are used with $=$, $\leq_{\text{triv}}$ and $=_{\text{triv}}$. Let $H(p) \in \mathbf{G}(p)$. For a k-ary lattice term t, the equivalence relation $t(\alpha_1|_{H(p)}, \dots, \alpha_k|_{H(p)}) \in L(p) \subseteq \mathrm{Equ}(V(p))$ will be denoted by $t|_{H(p)}$. For $t \in X$, a variable, $t|_{H(p)}$ has its previous meaning. By an (undirected) *$t|_{H(p)}$-path* we mean an (undirected) path U such that for every (undirected) edge (a, b) of U, $(a, b) \in t|_{H(p)}$. Similarly, for $n \geq 1$ and $\mu_1, \dots, \mu_n \in \mathrm{Equ}(V(p))$, an (undirected) path U is said to be an (undirected) *$\mu_1 \cup \dots \cup \mu_n$-path*, if $(a, b) \in \mu_1 \cup \dots \cup \mu_n$ for every (undirected) edge $(a, b) \in U$.

In what follows, the graph $H(p) = (V(p), E(p), \mathrm{col}) \in \mathbf{G}(p)$ is fixed. Let (a, r, b) be an edge of $H(p)$. Then the set $\{c : a \sqsubseteq c \sqsubseteq b\}$ of vertices determines a full subgraph denoted by $S(a, r, b)$. The left and right endpoint of $S(a, r, b)$ are a and b, respectively. If the color r is irrelevant, then we write $S(a, , b)$ instead of $S(a, r, b)$. Notice that $S(x_0, p, x_1)$ is $H(p)$, provided p is a join-irreducible term. It is clear from the construction that $S(a, r, b)$ is a graph. Moreover,

$$S(a, r, b) \cong H(r) \text{ for a (unique) } H(r) \in \mathbf{G}(r). \tag{4}$$

Notice that there is exactly one isomorphism between $H(r)$ and $S(a, r, b)$.

The following lemma is evident by the construction; we formulate it for later reference only.

Lemma 7 *Suppose that (a, r, b) is an edge of $H(p)$. Let $x, y \in V(p)$ such that x belongs to $S(a, r, b)$ but y does not. Let U be an undirected path in $H(p)$ from x to y. Then U goes through at least one of a and b.*

The following lemma is the heart our paper. Roughly saying, its first part states that the "outer world" does not disturb our equivalences inside $S(a, r, b)$.

Lemma 8 *Let t be a k-ary lattice term.*

(a) *If (a, r, b) is an edge of $H(p)$ and x and y are vertices of $S(a, r, b)$ then*

$$(x, y) \in t|_{H(p)} \quad \textit{iff} \quad (x, y) \in t|_{S(a,r,b)}.$$

(b) *Let x and y be vertices of $H(p)$. Then $(x, y) \in t|_{H(p)}$ iff there is an undirected $t|_{H(p)}$-path from x to y. In other words, $t|_{H(p)}$ is the equivalence generated by $t|_{H(p)} \cap E(p)$.*

Proof The proof is an induction on the length of t. The induction hypothesis is the *conjunction* of (a) and (b) for all terms t' shorter than t and for any p. (Notice that the induction would not work for (a) or (b) separately.) We assume that $x \neq y$. The "if" part of (a) and that of (b) are trivial (and, implicitly, will be used in the proof). So we will focus on the "only if" parts. Let $H_0(p)$, $H_1(p)$, $H_2(p)$, ... be the series of graphs that leads to $H(p)$ according to its inductive definition. We have to fix some notations according to p.

If p is join-irreducible, then let $m = \ell = c(1) = 1$, let $z_0 = x_0$, the left endpoint, $z_1 = x_1$, the right endpoint, and let $p_1 = p_{c(\ell)}$ stand for p.

If $p = \bigvee_{i \in F} p_i$ is join-reducible, then let $\{z_0 = x_0, z_1, \ldots, z_{m-1}, z_m = x_1\}$ be the vertex set and $\{(z_{i-1}, p_{c(i)}, z_i) : i = 1, 2, \ldots, m\}$ be the edge set of $H_0(p)$. Here all the $c(i)$ belong to F. If we wrote p_i in Figure 4 instead of t_i, then we would obtain an illustration for the case $F = \{1, 2, 3\}$. Clearly, there is a unique $\ell \in \{1, \ldots, m\}$ such that both a and b are vertices of $S(z_{\ell-1}, p_{c(\ell)}, z_\ell)$. Therefore, $S(a, r, b)$ is a full subgraph of $S(z_{\ell-1}, p_{c(\ell)}, z_\ell)$.

Case 1: $t = \beta \in X$ is a variable. Part (b) is evident. To prove (the "if" part of) (a), let us assume that $(x, y) \in \beta|_{H(p)}$. We also assume that $(a, b) \neq (x_0, x_1) = (z_0, z_n)$, because otherwise $S(a, r, b) = H(p)$, and there is nothing to prove.

Next, we assume that $(a,b) = (z_{\ell-1}, z_\ell)$. By the definition of $\beta|_{H(p)}$, there is a *shortest* undirected β-path in $H(p)$ that connects x and y. It follows from the structure of $H_0(p)$ (even without invoking Lemma 7) that any path exiting $S(a,r,b) = S(z_{\ell-1}, p_{c(\ell)}, z_\ell)$ at a can enter $S(a,r,b)$ again only at a, and the same holds for b. Hence our shortest β-path cannot exit $S(a,r,b)$ at all, and we conclude that $(x,y) \in \beta|_{S(a,r,b)}$.

Now that we have settled the easier subcases, we assume that $\{a,b\} \not\subseteq \{z_{\ell-1}, z_\ell\}$. Then there is a $j \geq 1$ such that a and b belong to $H_j(p)$, in fact to $S(z_{\ell-1}, p_{c(\ell)}, z_\ell) \cong H_j(p_{c(\ell)})$, but at least one of a and b is not in $H_{j-1}(p)$. Hence there is an edge (e,q,f) in $H_{j-1}(p)$, in fact in $S(z_{\ell-1}, p_{c(\ell)}, z_\ell)$, and there is an $s \in M(q)$ such that the edge (a,r,b) belongs to the s-arc glued to the edge (e,q,f) when $H_j(p)$ was obtained from $H_{j-1}(p)$, see Figure 5. This uniquely determined s-arc will be called the *supporting arc* of $S(a,r,b)$.

From the definition of an arc it follows that there is another r-colored edge of our s-arc, say (c,r,d). Notice that, opposed to Figure 5, $\{a,b,c,d\} \cap \{e,f\}$ is not necessarily empty. However, $\{a,b\} \cap \{c,d\} = \emptyset$ by the construction.

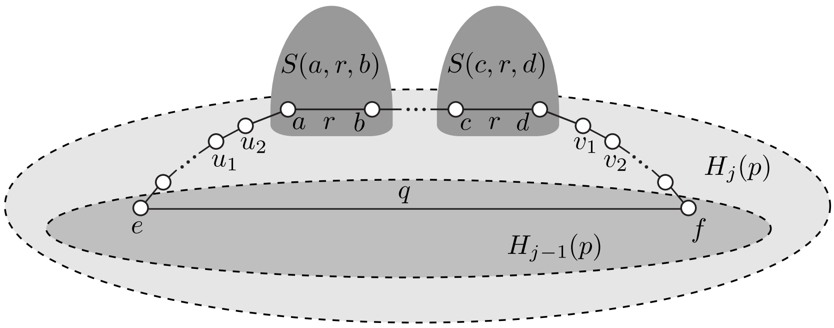

Figure 5: $S(a,r,b)$ and its supporting arc

Since $(x,y) \in \beta|_{H(p)}$, there is a *shortest* undirected β-path U in $H(p)$ from x to y. If U goes entirely in $S(a,r,b)$, then $(x,y) \in \beta|_{S(a,r,b)}$ and we are ready with this subcase. So assume that U leaves $S(a,r,b)$. Since U is a shortest path, we can assume by Lemma 7 that U leaves $S(a,r,b)$ at a and enters it again at b. (Interchanging a and b would make no difference in what follows.) Then, in the order given below, U must go through the vertices x, a, u_1, u_2, $\dots$, e of the supporting arc, then through f, $\dots$, v_2, v_1, d, c, $\dots$, b, y, see Figure 5. (Notice that these vertices are not necessarily consecutive vertices of U.)

Let W denote the segment of U between d and c. Since every path from d to c outside $S(c,r,d)$ should go through a, which would contradict to the assumption that U is the shortest path, we conclude that W goes entirely in $S(c,r,d)$. By stipulation (2), there is a graph isomorphism from $S(c,r,d)$ to $S(a,r,b)$. Replacing the "outer" a, $\dots$, e, f, $\dots$, b segment of U by the image of W, we obtain a shorter β-path from x to y, a contradiction. Hence $(x,y) \in \beta|_{S(a,r,b)}$, completing the case where t is a variable.

Case 2: t is meet-reducible with meetands $t_1, \dots, t_v$. We assume that the lemma is valid for the meetands $t_1, \dots, t_v$. Then part (a) of the lemma is clearly valid for t. To prove part (b), suppose that $(x, y) \in t|_{H(p)}$ and $x \neq y$.

Subcase 2.1: p is join-irreducible, that is, $m = 1$. Let j denote the smallest subscript such that both x and y belongs to $H_j(p)$; we will prove (b) for t by induction on j.

If $j = 0$, then $\{x, y\} = \{x_0, x_1\}$, whence (x, y) is an undirected edge, which is an undirected $t|_{H(p)}$-path. This settles the case $j = 0$.

Next, let $j > 0$, and assume that (b) holds for t and any two vertices from $H_{j-1}(p)$. We can assume that (x, y) is not an edge of $H(p)$. Let, say, x do not belong to $H_{j-1}(p)$. Then x belongs to an arc glued to $H_{j-1}(p)$, cf. Figure 5 with $x = a$. Suppose that e, the left endpoint of this arc, is *nearer* the edge $(x, r, b) = (a, r, b)$ than f. (The supporting arc consists of an even number of edges, so either e or f is strictly nearer.) According to the position of y, we have to distinguish two possibilities.

Sub-subcase 2.1.1: y is not on this arc. Let $i \in \{1, \dots, v\}$ be an arbitrary subscript. By the induction hypothesis, there is an undirected $t_i|_{H(p)}$-path U_i from x to y. This path leaves the arc at e or f.

We claim that there is an undirected $t_i|_{H(p)}$-path V_i from x to e. This is clear if U_i leaves the arc at e, so assume that it leaves the arc at f. Since e is nearer the edge (a, r, b) than f (in short, e is *near* and f is *far from* the edge (a, r, b)), each color on the arc between e and $a = x$ occurs between a and f. For example, let r' be the color of the edge (u_2, u_1), and also of the edge (v_1, v_2). Since U_i goes through v_1 and v_2, $(v_1, v_2) \in t_i|_{H(p)}$. Since part (a) is already valid for t_i, we get $(v_1, v_2) \in t_i|_{S(v_1, r', v_2)}$. It follows from stipulation (2) that

$$S(v_1, r', v_2) \cong S(u_2, r', u_1), \tag{5}$$

so $(u_2, u_1) \in t_i|_{S(u_2, r', u_1)}$, whence $(u_2, u_1) \in t_i|_{H(p)}$. This argument shows that the segment of the arc between e and $x = a$ is an undirected $t_i|_{H(p)}$-path, indeed.

This holds for all $i \in \{1, \dots, v\}$, and we conclude that there is an (undirected) $t|_{H(p)}$-path from x to $e \in H_{j-1}(p)$. Similarly, there is a $t|_{H(p)}$-path from y to a vertex $y' \in H_{j-1}(p)$. (Possibly, $y' = y$.) Since $(x, x'), (y, y') \in t|_{H(p)}$, the transitivity of $t|_{H(p)}$ implies that $(x', y') \in t|_{H(p)}$. By the induction hypothesis on j, there is an undirected $t|_{H(p)}$-path between x' and y'. Composing the three paths mentioned we obtain an undirected $t|_{H(p)}$-path from x to y, as requested.

Sub-subcase 2.1.2: y is on the same arc as x. Let $i \in \{1, \dots, v\}$, and consider a shortest (undirected) $t_i|_{H(p)}$-path U_i that connects x and y. Related to the arc, there are two possibilities for U_i. We say that it is a *detour*, if it consists of e, f, and all vertices of the arc that are not strictly between x and y. On the other hand, if U_i consists of all edges of the arc that are between x and y, then we say that U_i is a *straight path.* Clearly, U_i is either a detour or a straight path (but not both).

Similarly, there are two possibilities for the position of x and y; note that both possibilities can hold simultaneously. Namely, either x and y are *far* in the sense that each color occurring on the arc occurs between x and y, or x and y are *near* in the sense that each such color occurs not only between x and y.

Now assume that x and y are far. We claim that there is a $t_i|_{H(p)}$ detour connecting x and y. We have to investigate only the case when U_i is a straight path. Then, similarly to the argument above with (5), part (a) for t_i gives that every edge of the arc is a $t_i|_{H(p)}$-edge. By transitivity, $(e, f) \in t_i|_{H(p)}$. Hence the (unique) detour from x to y is an undirected $t_i|_{H(p)}$-path, indeed. This holds for all i, whence this detour is a $t|_{H(p)}$-path connecting x and y.

If x and y are near, then a straightforward analogous argument shows that the (unique) straight path from x to y is an undirected $t|_{H(p)}$-path.

Subcase 2.2: p is join-reducible, that is, $m \geq 2$. Firstly, assume that x and y belong to the same subgraph $S(z_{\ell-1}, p_{c(\ell)}, z_\ell)$. For all $i \in \{1, \ldots, v\}$, $(x, y) \in t_i|_{S(z_{\ell-1}, p_{c(\ell)}, z_\ell)}$ by part (a) of the lemma. Since $p_{c(\ell)}$ is join-irreducible and we have $S(z_{\ell-1}, p_{c(\ell)}, z_\ell) \cong H(p_{c(\ell)})$ for an appropriate $H(p_{c(\ell)}) \in \mathbf{G}(p_{c(\ell)})$, the previous case implies the existence of a $t|_{S(z_{\ell-1}, p_{c(\ell)}, z_\ell)}$ path from x to y. It is clearly a $t|_{H(p)}$-path.

Secondly, assume that x belongs to the subgraph $S(z_{\ell-1}, p_{c(\ell)}, z_\ell)$ and y belongs to $S(z_{h-1}, p_{c(h)}, z_h)$. Let, say, $\ell < h$. We know that there are shortest $t_i|_{H(p)}$-paths U_i from x to y for $i \in \{1, \ldots, v\}$. There can be no detours now, so all these paths go through $z_\ell, z_{\ell+1}, \ldots, z_{h-1}$. This holds for all $i \in \{1, \ldots, v\}$, whence (x, z_ℓ), $(z_\ell, z_{\ell+1})$, $\ldots$, (z_{h-1}, y) belong to $t|_{H(p)}$. Since the components of each of these pairs belong to the same subgraph, the previous case yields that these components can be connected by $t|_{H(p)}$-paths. Putting these paths together, we obtain a $t|_{H(p)}$-path from x to y.

Case 3: t is join-reducible with joinands $t_1, \ldots, t_v$. Suppose that the lemma is valid for these joinands. Since $t_i|_{H(p)}$-paths are $t|_{H(p)}$-paths as well, part (b) of the lemma is evident.

The argument for part (a) is similar to the case when t was a variable, so we will use the notations introduced in connection with Figure 5. In particular, x and y are vertices of $S(a, r, b)$ and $(x, y) \in t|_{H(p)} = t_1|_{H(p)} \vee \cdots \vee t_v|_{H(p)}$. Using the description of joins in $\mathrm{Equ}(V(p))$ and then the induction hypothesis for the t_i, we obtain a shortest undirected $t_1|_{H(p)} \cup \cdots \cup t_v|_{H(p)}$-path U connecting x and y. We want to show that U goes entirely in $S(a, r, b)$.

This is evident if (a, b) is an edge of $H_0(p)$, that is, it is of the form $(z_{\ell-1}, z_\ell)$. So, assume that (a, b) is not an edge of $H_0(p)$ and, by way of contradiction, assume that U exits $S(a, r, b)$. Then a segment of U connects c and d within $S(c, r, d)$. Each edge of this segment is collapsed by some $t_i|_{H(p)}$, whence by $t_i|_{S(c,r,d)}$ according to the induction hypothesis. Using the isomorphism between $S(c, r, d)$ and $S(a, r, b)$, we obtain a *shorter* path from a to b within $S(a, r, b)$ whose edges are collapsed by appropriate $t_i|_{S(a,r,b)}$, whence by $t_i|_{H(p)}$.

This contradiction shows that U goes in $S(a, r, b)$, indeed. By the induction hypothesis, if an edge of U is collapsed by $t_i|_{H(p)}$ then it is collapsed by $t_i|_{S(a,r,b)}$, and therefore by $t|_{S(a,r,b)}$. Finally, $(x, y) \in t|_{S(a,r,b)}$ follows by transitivity. □

The next lemma will obviously imply Theorem 3 and Proposition 1.

Lemma 9 *The k-pointed lattice $L(p)$ defined by formula (3) is a p-lattice. Moreover, the following three conditions are equivalent for any k-ary lattice term q:*

(a) $p \le_{\text{triv}} q$;

(b) $p(\alpha_1|_{H(p)}, \ldots, \alpha_k|_{H(p)}) \le q(\alpha_1|_{H(p)}, \ldots, \alpha_k|_{H(p)})$ *in* $L(p)$;

(c) $(x_0, x_1) \in q(\alpha_1|_{H(p)}, \ldots, \alpha_k|_{H(p)})$.

Proof (a) implies (b) trivially. An easy induction on the length of p gives $(x_0, x_1) \in p(\alpha_1|_{H(p)}, \ldots, \alpha_k|_{H(p)}) = p|_{H(p)}$, whence (b) implies (c).

Next, suppose (c), let L be an arbitrary lattice, and let $\beta_1, \ldots, \beta_k \in L$. We know from Jónsson [10] that each lattice has a type 3 representation, see also Theorem IV.4.4 in Grätzer [8]. Hence we can assume that L is a sublattice of some $\mathrm{Equ}(Y)$ and $\gamma \vee \delta = \gamma \circ \delta \circ \gamma \circ \delta$ holds for any $\gamma, \delta \in L$. Let $(y_0, y_1) \in p(\beta_1, \ldots, \beta_k)$. A straightforward induction on the length of p shows the existence of a map $\varphi\colon V(p) \to Y$ such that $x_0 \mapsto y_0$, $x_1 \mapsto y_1$, and for each α_i-edge (u, α_i, v) of $H(p)$, we have $(u\varphi, v\varphi) \in \beta_i$. The same kind of induction on the length of q shows that, for $a, b \in V(p)$, if $(a, b) \in q|_{H(p)}$, then $(a\varphi, b\varphi) \in q(\beta_1, \ldots, \beta_k)$. In particular, $(y_0, y_1) \in (x_0\varphi, x_1\varphi) \in q(\beta_1, \ldots, \beta_k)$. Hence $p \le q$ holds in L, so $p \le_{\text{triv}} q$. □

Corollary 10 *Let (a, r, b) be an edge of $H(p) \in \mathbf{G}(p)$, and let t be an arbitrary k-ary lattice term. Then*

(a) $r|_{H(p)}$ *is the smallest element of* $L(p)$ *that collapses* a *and* b;

(b) $(a, b) \in t|_{H(p)}$ *if and only if* $r \le_{\text{triv}} t$;

(c) $r|_{H(p)} \le t|_{H(p)}$ *if and only if* $r \le_{\text{triv}} t$.

Proof By (4), there is a (unique) graph $H(r) \in \mathbf{G}(r)$ such that $H(r) \cong S(a, r, b)$. Since $r \le_{\text{triv}} r$, we conclude $(x_{0,H(r)}, x_{1,H(r)}) \in r|_{H(r)}$ by Lemma 9. Hence $(a, b) \in r|_{S(a,r,b)}$, and Lemma 8(a) gives $(a, b) \in r|_{H(p)}$.

Next, assume that $(a, b) \in t|_{H(p)}$. We obtain from Lemma 8(a) that $(a, b) \in t|_{S(a,r,b)}$. Hence $(x_{0,H(r)}, x_{1,H(r)}) \in t|_{H(r)}$, so $r \le_{\text{triv}} t$ by Lemma 9. This proves part (b) and completes the proof of part (a). Finally, (c) is an evident consequence of (a) and (b). □

Corollary 11 *Suppose $\mu \in L(p)$, (a, r, b) is an edge of $H(p)$, and t is a k-ary lattice term. Then*

(a) *it depends only on* r *if* $(a, b) \in \mu$;

(b) $\mu = \bigvee\{s|_{H(p)} : s \in C(p)$ *and all* s*-colored edges are collapsed by* $\mu\}$.

(c) $t|_{H(p)} = \bigvee\{s|_{H(p)} : s \in C(p)$ *and* $s \le_{\text{triv}} t\}$.

Proof Since μ is of the form $t|_{H(p)}$, part (a) follows from Corollary 10(b).

Let $B = \{s \in C(p) : \text{all } s\text{-colored edges are collapsed by } \mu\}$ and $\nu = \bigvee_{s \in B} s|_{H(p)}$. Suppose (c, s, d) is an edge with $(c, d) \in \mu = t|_{H(p)}$. Then $s \in B$ by part (a), and $(c, d) \in s|_{H(p)}$ by Corollary 10(a). Hence $(c, d) \in \nu$. Therefore, Lemma 8(b) implies $\mu = t|_{H(p)} \leq \nu$. Conversely, Corollary 10(b) yields that $s \leq_{\text{triv}} t$ for every $s \in B$. Hence $\nu \leq t|_{H(p)} = \mu$, proving part (b).

Finally, part (c) is a consequence of part (b) and Corollary 10(b). □

The following two corollaries (and the dual of the second one) say that free lattices satisfy Whitman's condition. Their original proof in [13] is a bit lengthy. Based on A. Day [5], the approach of Freese, Ježek, Nation [6] to Whitman's condition is shorter. Now, armed with the basic properties of $L(p)$, we are going to give an even shorter proof. Since it is visual, it reveals some new ingredients from the underlying reasons.

Corollary 12 (Whitman [13]) *Let p be a meet-reducible lattice term with with meetands p_1, …, p_u, and let q be a join-reducible lattice term with joinands q_1, …, q_v. Assume that $p \leq_{\text{triv}} q$. Then either $p_i \leq_{\text{triv}} q$ for some $i \in \{1, \dots, u\}$ or $p \leq_{\text{triv}} q_j$ for some $j \in \{1, \dots, v\}$.*

Proof Lemma 9 yields that $(x_0, x_1) \in q|_{H(p)} = q_1|_{H(p)} \vee \dots \vee q_v|_{H(p)}$. Hence there exists a *shortest* undirected $q_1|_{H(p)} \cup \dots \cup q_v|_{H(p)}$-path U that connects x_0 and x_1.

Firstly, if U is of length 1, then $p \leq_{\text{triv}} q_j$ for some j by Lemma 9.

Secondly, if $\text{length}(U) \geq 2$, then U goes through all vertices of a unique p_i-arc glued to $H_0(p)$. Hence U goes within $H(p_i)$. Let (c, s, d) be an edge of U. Then $(c, d) \in q|_{H(p)}$. Using Lemma 8(a) twice, we get $(c, d) \in q|_{S(c,s,d)}$ and $(c, d) \in q|_{H(p_i)}$. By transitivity, $(x_0, x_1) \in q|_{H(p_i)}$. Hence $p_i \leq_{\text{triv}} q$ by Lemma 9. □

Corollary 13 *Let $p = \bigwedge_{i=1}^{u} p_i$ and $q = \bigvee_{i=1}^{v} q_i$ as in the previous corollary, and let α_i be a variable. Then*

- *if $\alpha_i \leq_{\text{triv}} q$ then $\alpha_i \leq_{\text{triv}} q_j$ for some $j \in \{1, \dots, v\}$;*
- *if $p \leq_{\text{triv}} \alpha_i$ then $p_j \leq_{\text{triv}} \alpha_i$ for some $j \in \{1, \dots, u\}$.*

To demonstrate the usefulness of test lattices, we prove the two parts of this corollary separately even if each of them implies the other by the duality principle.

Proof For the first part, let $p' = \alpha_i$ and $H(p') \in \mathbf{G}(p')$. Since $|V(H(p'))| = |L(p')| = 2$ and $1_{L(p')}$ is join-irreducible, we obtain from $(x_0, x_1) \in 1_{L(p')} = p'|_{H(p')} \leq q|_{H(p')} = q_1|_{H(p')} \vee \dots \vee q_v|_{H(p')}$ that $(x_0, x_1) \in q_j|_{H(p')}$ for some j. Hence $\alpha_i \leq_{\text{triv}} q_j$ by Lemma 9.

For the second part, take a shortest α_i-path U connecting x_0 and x_1 in $H(p)$. If $\text{length}(U) = 1$, then α_i equals a meetand p_j of p, whence $p_j \leq_{\text{triv}} \alpha_i$. If $\text{length}(U) \geq 2$, then U goes within some $H(p_j)$, and $p_j \leq_{\text{triv}} \alpha_i$ by Lemma 9. □

For a congruence Θ of a k-pointed lattice $(L;\vec{d})$, we will use the notation $\vec{d}/\Theta = (d_1/\Theta, \dots, d_k/\Theta)$. Let us call Θ a *p-preserving congruence*, if $(c, p(\vec{d})) \in \Theta$ holds for no $c < p(\vec{d})$. The following lemma implies Corollary 4; we formulate this lemma for a later reference. By homomorphisms we still mean k-pointed lattice homomorphisms, and isomorphisms are particular cases.

Lemma 14 *Let $(L,\vec{d})$ be a p-lattice, and let Θ be a congruence of $(L;\vec{d})$.*

- *$(L/\Theta;\vec{d}/\Theta)$ is a p-lattice iff Θ is p-preserving.*
- *There exists an optimal p-lattice. It is finite, and it is unique up to isomorphism.*
- *$(L;\vec{d})$ is an optimal p-lattice iff 0 is the only p-preserving congruence of $L(p)$.*

Proof Assume that Θ is not p-preserving, and choose an element $c = q(\vec{d})$ such that $c < p(\vec{d})$ and $(c, p(\vec{d})) \in \Theta$. Then $p(\vec{d}/\Theta) \le q(\vec{d}/\Theta)$, for they are equal, but $p \not\le_{\text{triv}} q$, so L/Θ is not a p-lattice. Conversely, suppose that Θ is p-preserving and $p(\vec{d}/\Theta) \le q(\vec{d}/\Theta)$. Then $p(\vec{d}/\Theta) \wedge q(\vec{d}/\Theta) = p(\vec{d}/\Theta)$ gives $(p(\vec{d}) \wedge q(\vec{d}), p(\vec{d})) \in \Theta$. Using that Θ is p-preserving, we get $p(\vec{d}) \wedge q(\vec{d}) = p(\vec{d})$. This means that $p(\vec{d}) \le q(\vec{d})$ in L, whence $p \le_{\text{triv}} q$, proving the first part.

Let $F = [d_1, \dots, d_k]$ be the free lattice generated by $\{d_1, \dots d_k\}$. Then $(F;\vec{d})$ is a p-lattice, whence its smallest congruence is p-preserving. Since the (nonempty) join of all p-preserving congruences of $(F;\vec{d})$ is clearly p-preserving by Lemma III.1.3 from Grätzer [8], $(F;\vec{d})$ has a largest p-preserving congruence Ψ. By the first part of the Lemma, $(K,\vec{d}) := (F/\Psi;\vec{d}/\Psi)$ is a p-lattice.

Let $(M;\vec{d})$ be another p-lattice. Let φ denote the surjective lattice homomorphism $\varphi\colon F \to M$, $d_1 \mapsto d_1, \dots, d_k \mapsto d_k$, that is, the unique k-pointed lattice homomorphism from $(F;\vec{d})$ to $(M;\vec{d})$. Clearly, $\operatorname{Ker}\varphi \subseteq \Psi$, whence $(K;\vec{d}) \cong (F/\Psi;\vec{d}/\Psi)$ is a homomorphic image of $(M,\vec{d}) \cong (F/\operatorname{Ker}\varphi;\vec{d}/\operatorname{Ker}\varphi)$. Hence $(K;\vec{d})$ is an optimal p-lattice. It is finite by Theorem 3. Its uniqueness is an evident consequence of finiteness. This proves the second part.

To prove the third part, let Θ be a p-preserving congruence of an *optimal* p-lattice $(L;\vec{d})$. By the first part, $(L/\Theta;\vec{d}/\Theta)$ is again a p-lattice. So, $(L;\vec{d})$ is a homomorphic image of $(L/\Theta;\vec{d}/\Theta)$, and the finiteness of L implies $\Theta = 0$.

Conversely, assume that 0 is the only p-preserving congruence of a p-lattice $(L;\vec{d})$. Consider the (unique) homomorphism $\varphi\colon (L;\vec{d}) \to (K;\vec{d})$. Since

$$(L;\vec{d})/\operatorname{Ker}\varphi \cong (K;\vec{d})$$

is a test lattice, $\operatorname{Ker}\varphi$ is p-preserving by the first part. Hence $\operatorname{Ker}\varphi = 0$ yields that φ is an isomorphism. This implies that $(L;\vec{d})$ is an optimal p-lattice. □

Let $H(p) \in \mathbf{G}(p)$, and let $U = (x_0 = a_0, a_1, a_2, \dots, a_n = x_1)$ be a *directed* path in $H(p)$. We say that U is a *uniform path*, if the following condition holds: for any $0 \le i_1 < i_2 < i_3 < i_4 \le n$ such that (a_{i_1}, a_{i_2}) and (a_{i_3}, a_{i_4}) are edges of the same color r, the unique isomorphism $S(a_{i_1}, r, a_{i_2}) \to S(a_{i_3}, r, a_{i_4})$, see (4), sends the segment of U between a_{i_1} and a_{i_2} onto the segment of U between a_{i_3} and a_{i_4}.

Lemma 15 *Let U be a uniform path as above and let $\{r_1, \ldots, r_m\}$ be the set of colors of edges of U. Then* $\text{length}(r_1 \vee \cdots \vee r_m) \le \text{length}(p)$.

Proof We use induction on length(p). If p is a variable or $n = \text{length}(U) = 1$, then the statement is evident. If p is join-reducible, then $n > 1$ and the induction step is straightforward. If p is meet-reducible and $n > 1$, then there is an $s \in M(p)$, see (1), such that U includes the vertices of the s-arc glued to $H_0(p)$, and the induction step is straightforward again. □

Proof of Theorem 5 According to Lemma 14, it suffices to show that Θ is not p-preserving for any nontrivial congruence Θ of $L(p)$. Since Θ is nontrivial, $\mu < \nu$ and $(\mu, \nu) \in \Theta$ hold for some $\mu, \nu \in L(p)$. In virtue of Lemma 8(b), there is an edge (a, r, b) with $(a, b) \in \nu \setminus \mu$. Let $\eta = \mu \cap r|_{H(p)}$. Corollary 11 implies that $r|_{H(p)} = r|_{H(p)} \wedge \nu$. Hence

$$\eta < r|_{H(p)}, \quad (\eta, r|_{H(p)}) \in \Theta \text{ and } (a, b) \in r|_{H(p)} \setminus \eta. \tag{6}$$

Let us fix an $r \in C(p)$ with *maximal length* such that (6) holds with an appropriate edge (a, r, b) and an $\eta \in L(p)$. According to Corollary 11(b), there are $t_1, \ldots, t_u \in C(p)$ such that

$$\eta = t_1|_{H(p)} \vee \cdots \vee t_u|_{H(p)}. \tag{7}$$

Let j denote the unique subscript from $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ such that (a, r, b) is an edge of $H_j(p)$ but not of $H_{j-1}(p)$.

We have to consider several cases.

Case 1: $j > 0$. Then there is a meet-reducible $q \in C(p)$, an edge (e, q, f) of $H_{j-1}(p)$, and a meetand s of q such that

$$s = r \vee t_{u+1} \vee \cdots \vee t_{u+v} \in M(q). \tag{8}$$

In particular,

$$s|_{H(p)} = r|_{H(p)} \vee t_{u+1}|_{H(p)} \vee \cdots \vee t_{u+v}|_{H(p)}. \tag{9}$$

Notice that $v \ge 1$, and the situation is similar to that of Figure 5. Let

$$\delta := \eta \vee t_{u+1}|_{H(p)} \vee \cdots \vee t_{u+v}|_{H(p)} = t_1|_{H(p)} \vee \cdots \vee t_{u+v}|_{H(p)}. \tag{10}$$

Then (6), (9) and (10) yield that $(\delta, s|_{H(p)}) \in \Theta$ and $\delta \le s|_{H(p)}$. Since

$$\delta \not< s|_{H(p)} \text{ by } \text{length}(r) < \text{length}(s), \tag{11}$$

we conclude that

$$\delta = s|_{H(p)}.$$

Since $(e,f) \in q|_{H(p)}$ by Corollary 10(a) and, clearly, $q \le_{\rm triv} s$, we obtain that

$$(e,f) \in s|_{H(p)} = t_1|_{H(p)} \vee \cdots \vee t_{u+v}|_{H(p)} = (t_1 \vee \cdots \vee t_{u+v})|_{H(p)}. \tag{12}$$

Since $t_1, \ldots, t_{u+v} \in C(p)$, (12) and Corollary 10(b) imply

$$t_1 \vee \cdots \vee t_{u+v} \le_{\rm triv} s. \tag{13}$$

Let $H(q) := S(e,q,f)$. Then $H(q) \in \mathbf{G}(q)$ by (4), and

$$(e,f) \in (t_1 \vee \cdots \vee t_{u+v})|_{H(q)} = t_1|_{H(q)} \vee \cdots \vee t_{u+v}|_{H(q)} \tag{14}$$

follows from (12) and Lemma 8(a). Hence, by Lemma 8(b), there is a $t_1|_{H(q)} \cup \cdots \cup t_{u+v}|_{H(q)}$-path U in $H(q) = S(e,q,f)$ that connects e and f. We can assume that U goes through each vertex of $H(q)$ at most once. Then a trivial induction on length(q) shows that U is a *directed* path. Another trivial induction on length(q), based on (4), yields that U can be chosen to be uniform. Let (x,c,y) be an edge of U. Then $(x,y) \in t_i|_{H(q)}$ for some i. Hence $(x,y) \in t_i|_{H(p)}$ by the (trivial direction of) Lemma 8(a).

This shows that U is a uniform $t_1|_{H(p)} \cup \cdots \cup t_{u+v}|_{H(p)}$-path from e to f; in fact, we assume that U is the shortest uniform path with this property.

Subcase 1.1: U consists of a single edge. Then $(e,f) \in t_i|_{H(p)}$ and Corollary 10(b) yield that $q|_{\le_{\rm triv}} t_i$ for some $i \in \{1, \ldots, u+v\}$.

Firstly, assume that $i \le u$. Then $q|_{H(p)} \le t_i|_{H(p)} \le \eta \le r|_{H(p)}$. Hence Corollary 10(c) implies $q \le_{\rm triv} r$. Let q' denote the lattice term that we obtain from q by replacing its meetand s with r. Then $q \le_{\rm triv} r \le_{\rm triv} s$ implies $q' =_{\rm triv} q$. Since length$(r) <$ length(s), we see that length$(q') <$ length(q). So, q is not a canonical term. This is a contradiction, for all subterms of the canonical p are canonical.

Secondly, assume that $u < i \le u+v$. Then $q \le_{\rm triv} t_i \le_{\rm triv} s$, like above. Hence, using t_i instead of r, we can derive the same contradiction.

Subcase 1.2: U consists of at least two edges. Then there is an $s' \in M(q)$, see (1), such that U goes through all the vertices of the s'-arc that was glued to $H_{j-1}(p)$.

Sub-subcase 1.2.1: s' and s are distinct. Let

$$z_0 = e, z_1, \ldots, z_{n-1}, z_n = f \quad \text{and} \quad (z_0, t'_1, z_1), \ldots, (z_{n-1}, t'_n, z_n)$$

be the vertices and the edges of the s'-arc, respectively. Since U goes through z_{i-1} and z_i,

$$(z_{i-1}, z_i) \in t_1|_{H(p)} \vee \cdots \vee t_{u+v}|_{H(p)} = \delta = s|_{H(p)}$$

holds for $i \in \{1, \ldots, n\}$. By Corollary 10(b), $t'_i \le_{\rm triv} s$ for all i, which yields that $s' = t'_1 \vee \cdots \vee t'_n \le_{\rm triv} s$. This is a contradiction, for the canonical term q cannot have two trivially comparable meetands.

Sub-subcase 1.2.2: s' and s are the same. Then a section W of U, which is a uniform path again, connects a and b. Let $t'_1, \ldots, t'_w$ be the colors of the edges of W. By Corollary 10(b),

$$\forall j \in \{1, \ldots, w\} \ \exists i \in \{1, \ldots, u+v\} \text{ such that } t'_j \leq_{\text{triv}} t_i. \tag{15}$$

Corollary 10(a), applied to the edges of W, and transitivity imply

$$(a, b) \in t'_1|_{H(p)} \vee \cdots \vee t'_w|_{H(p)} = (t'_1 \vee \cdots \vee t'_w)|_{H(p)}.$$

Hence Corollary 10(b) implies

$$r \leq_{\text{triv}} t'_1 \vee \cdots \vee t'_w. \tag{16}$$

This together with (8) yields that $s \leq_{\text{triv}} t'_1 \vee \cdots \vee t'_w \vee t_{u+1} \vee \cdots \vee t_{u+v}$. Conversely, $t'_1 \vee \cdots \vee t'_w \vee t_{u+1} \vee \cdots \vee t_{u+v} \leq^{(15)}_{\text{triv}} t_1 \vee \cdots \vee t_{u+v} \leq^{(13)}_{\text{triv}} s$. Hence

$$s =_{\text{triv}} t'_1 \vee \cdots \vee t'_w \vee t_{u+1} \vee \cdots \vee t_{u+v}. \tag{17}$$

If $i = i(j)$ belonged to $\{1, \ldots, u\}$ in (15) for each j, then

$$r|_{H(p)} \leq^{(16)} (t'_1 \vee \cdots \vee t'_w)|_{H(p)} \leq (t_1 \vee \cdots \vee t_u)|_{H(p)} =^{(7)} \eta$$

would contradict (6). Hence, by (15), there is a j, say $j = 1$, such that $t'_1 \leq_{\text{triv}} t_i$ holds for some $i \in \{u+1, \ldots, u+v\}$. Let $g = t'_2 \vee \cdots \vee t'_w \vee t_{u+1} \vee \cdots \vee t_{u+v}$. We see by (17) that $s =_{\text{triv}} g$. However, (8) together with $\text{length}(t'_1 \vee \cdots \vee t'_w) \leq^{(\text{L15})} \text{length}(r)$ yields that $\text{length}(g) < \text{length}(s)$. This is a contradiction, because s, as a subterm of p, is canonical.

Case 2: $j = 0$. Firstly, assume p is join-irreducible. Then $H_0(p)$ consists of a single p-colored edge, r coincides with p, whence Θ is not p-preserving, indeed.

Secondly, assume that p is join-reducible. With the temporary notations $s' = s := p$, $e := x_0$ and $f := x_1$, the argument for Sub-subcase 1.2.2 works almost the same way as previously. The only difference is that, instead of (11), we say that

- either $\delta \not< s|_{H(p)}$ and we derive a contradiction the same way as before,
- or $\delta < s|_{H(p)} = p|_{H(p)}$, whence Θ is not p-preserving, indeed.

(Since (e, f) is not an edge now, (11) in itself would not work.) □

Proof of Theorem 6 We can assume that a p is not a join-free term, because otherwise $|L(p)| = 2$ and there is nothing to prove.

We claim that $p|_{H(p)}$ is a join-irreducible element of $L(p)$. By way of contradiction, suppose that there are terms h_1 and h_2 such that $h_1|_{H(p)} < p|_{H(p)}$, $h_2|_{H(p)} < p|_{H(p)}$ but $h_1|_{H(p)} \vee h_1|_{H(p)} = p|_{H(p)}$. Similarly to (and even easier than) the argument right after (14), we conclude that there is a uniform $h_1|_{H(p)} \cup h_2|_{H(p)}$-path U connecting x_0 and x_1. Since $\text{length}(U) = 1$ would imply $p \leq_{\text{triv}} h_i$ for some $i \in \{1, 2\}$ by Corollary 10(b), we obtain that $\text{length}(U) > 1$.

Hence there is an $s \in M(p)$ such that U goes through the vertices of the s-arc glued to $H_0(p)$. Let $s = \bigvee_{i=1}^{n} t_i$. Since U is also a $p|_{H(p)}$-path, we get $t_i \leq_{\mathrm{triv}}^{(\mathrm{C10})} p$ for $i = 1, \dots, n$. Hence $s \leq_{\mathrm{triv}} p$. Since s is a meetand of p, we have $p \leq_{\mathrm{triv}} s$. Since p is canonical, p coincides with s, which contradicts the assumption that p is a join-irreducible term.

This proves that $p|_{H(p)}$ is join-irreducible in $L(p)$. It is not the 0 of $L(p)$, since $(x_0, x_1) \in p|_{H(p)}$ by Corollary 10(a). Hence $p|_{H(p)}$ has a unique lower cover $p_*|_{H(p)}$. Since congruence classes are intervals and $L(p)$ is optimal by Theorem 5, it follows by Lemma 14 that each non-zero congruence of $L(p)$ collapses $p|_{H(p)}$ and $p_*|_{H(p)}$. Thus, $L(p)$ is subdirectly irreducible. □

It is trivial to check that, for any ternary term q, if q is shorter than $p^{\diamondsuit}$ of Exercise 2, then the identity $p = q$ fails even in the free modular lattice on three generators. Hence $p^{\diamondsuit}$ is a join-irreducible canonical term. It is also trivial to verify that $|L(p^{\diamondsuit})| > 5$. Notice that even Figure 6, which is a useful illustration for test lattices, was very easy to construct. Hence the following proposition clearly solves Exercise 2. In Proposition 16, K will be a lattice in the usual sense while $(L(p); \vec{d})$, the p-lattice, is a k-pointed lattice.

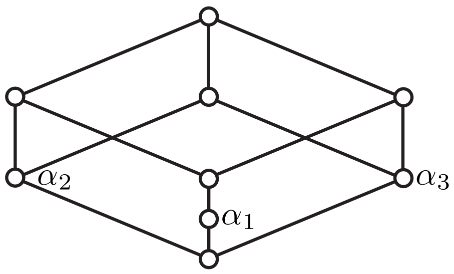

Figure 6: The test lattice $L(p^{\diamondsuit})$

Proposition 16 *Let p be a join-irreducible canonical k-ary lattice term, and let K be a lattice with $|K| < |L(p)|$. Then there exists a k-ary lattice term q such that $p \leq q$ is a nontrivial lattice identity that holds in K.*

Proof Let $n = |K|$. There are n^k ways to make K into a k-pointed lattice $(K; \vec{d})$ by selecting k elements in K. Let $(G; \vec{d})$ be the direct product of all these (n^k many) k-pointed lattices.

Assume that the proposition fails for K. Then $(G; \vec{d})$ is a p-lattice. The (unique) optimal p-lattice is a homomorphic image of $(G; \vec{d})$ by Lemma 14. But, by Theorem 5, the optimal p-lattice is $(L(p); \vec{d})$. Therefore, $L(p)$, as a lattice without constants, belongs to the variety generated by K. Since $L(p)$ is subdirectly irreducible by Theorem 6, the famous $\mathbf{HSP} = \mathbf{P}_s\mathbf{HSP}_u$ theorem of B. Jónsson [11] gives that $L(p)$ is a homomorphic image of a sublattice of K. This contradicts $|K| < |L(p)|$. □

4 Historical remarks

Graphs similar to those here were formerly useful in [1], [2], [4], M. Haiman [9] and P. Lipparini [12]. Even one of the efficient known algorithms for the word problem of lattices is due to graphs, see [3]. (For other algorithms, see also Section XI.8 of Freese, Ježek, Nation [6]). In fact, [3] gives the main motivation to the present work: if graphs are appropriate to solve the word problem, then why not use them for other purposes? However, the mentioned similarity is limited, because our graphs here have more edges than their precursors. In fact, finding the right amount of edges was the main step towards the present approach.

The results of this paper were presented at the conferences organized by the University of Nov Sad and the Technical University of Košice, respectively. It has appeared since then that our approach overlaps Freese, Ježek and Nation [6] more than previously recognized. Since the concepts and the methods of [6] are very different from ours and the counterparts of our results are sometimes only implicitly given in [6], it is reasonable to give a short comparison below.

If we do not assume that p is canonical, then, generally, $L(p)$ does not occur in the book [6]. So, in what follows, let as assume that p' is a *canonical* lattice term.

Using Theorem 3.12 of [6] (in short, Thm. [6].3.12), it is easy to see that $J(p') = J^*(p')$ of [6] is the same as our $C(p')$. Then Cor. [6].3.18 together with Corollary 11(c) gives that $L^\vee(p')$ coincides with our $L(p')$, whence it is our $K(p')$ by Theorem 5. This shows that each optimal test lattice $K(p')$ has been constructed in [6]. This shows also that Theorem 6 is included in Thm. [6].3.24.

The result that $L^\vee(p')$ is a p'-lattice can be easily extracted from [6] in the following way. By the second and third sentences in the proof of Thm. [6].3.15, f in Cor. [6].3.18 is a contraction that acts identically on $L^\vee(p')$, which is a join-subsemilattice of the free lattice $FL(\alpha_1, \dots, \alpha_k)$. Hence $p' \in FL(\alpha_1, \dots, \alpha_k)$ is the least preimage of $p' \in L^\vee(p')$. So, $f(p') \le f(q)$ implies $p' \le_{\text{triv}} q$, whence $L^\vee(p')$ is a p'-lattice.

It is also possible to extract from [6] that $L^\vee(p')$ is an *optimal* p'-lattice; however, this would require a deeper look into the book, so the details are omitted.

In connection with Theorems 5 and 6, we notice that the name "canonical term" in the present paper means only a *shortest* term, which trivially exists. Opposed to [6], we do not use Whitman's non-trivial theorem on its uniqueness, see [13].

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