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Frankl's Conjecture for Large Semimodular and Planar Semimodular Lattices*

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Abstract

A lattice L is said to satisfy (the lattice theoretic version of) Frankl's conjecture if there is a join-irreducible element $f \in L$ such that at most half of the elements x of L satisfy $f \leq x$. Frankl's conjecture, also called as union-closed sets conjecture, is well-known in combinatorics, and it is equivalent to the statement that every finite lattice satisfies Frankl's conjecture.

Let m denote the number of nonzero join-irreducible elements of L. It is well-known that L consists of at most 2^m elements. Let us say that L is large if it has more than $5 \cdot 2^{m-3}$ elements. It is shown that every large semimodular lattice satisfies Frankl's conjecture. The second result states that every finite semimodular planar lattice L satisfies Frankl's conjecture. If, in addition, L has at least four elements and its largest element is join-reducible then there are at least two choices for the above-mentioned f.

Key words: Union-closed sets; Frankl's conjecture; lattice, semimodularity; planar lattice.

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Given an m-element finite set $A = \{a_1, \dots, a_m\}$, $m \geq 3$, a *family* (or, in other words, a set) $\mathcal{F}$ of at least two subsets of A, i.e. $\mathcal{F} \subseteq P(A)$, is called a *union-closed family* (over A) if $X \cup Y \in \mathcal{F}$ whenever $X, Y \in \mathcal{F}$. It was Peter Frankl in 1979 (cf. Frankl [9]) who formulated the following conjecture, now called as *Frankl's conjecture* or *the union-closed sets conjecture*: if $\mathcal{F}$ is as above then there exists an element of A which is contained in at least half of the members of $\mathcal{F}$. In spite of at least three dozen papers, cf. the bibliography given in [8], this conjecture is still open.

Now let L be a finite lattice. As usual, the set of its nonzero join-irreducible elements will be denoted by $J(L)$. We say that L satisfies (the lattice theoretic version of) Frankl's conjecture if $|L| = 1$ or there is an $f \in J(L)$ such that for the principal filter $\uparrow f = \{x \in L \colon f \leq x\}$ we have $|\uparrow f| \leq |L|/2$. Stanley [17] and Poonen [14] or Abe and Nakano [3] have shown that (the original) Frankl's conjecture is true if and only if all finite lattices satisfy (the lattice theoretic) Frankl's conjecture. (For details one can also see [6].) This fact has initiated a series of lattice theoretical results given by Abe and Nakano [1], [2], [3], [4], Herrmann and Langsdorf [13], and Reinhold [15], and two combinatorial results achieved by means of lattices, cf. [6] and [8]. In particular, lower semimodular lattices satisfy Frankl's conjecture by [15], and the method of [15] makes it clear that the situation for (upper) semimodular lattices is much harder. In fact, it is (and it remains) unknown if semimodular lattices satisfy Frankl's conjecture. The goal of the present paper is to present two subclasses of the class of finite semimodular lattices such that every lattice L in these subclasses satisfies Frankl's conjecture; in fact, L usually satisfies the conjecture in a bit stronger form.

For elements x and y of a lattice L, let $x \preceq y$ denote the "covers or equals" relation. That is, $x \preceq y$ iff $x \leq y$ and there is no $z \in L$ with $x < z < y$. Recall that L is called (upper) *semimodular* if, for any $a, b, c \in L$, $a \preceq b$ implies $a \vee c \preceq b \vee c$. Let $J(L)$ denote the set of non-zero join-irreducible elements of L, and let $m = |J(L)|$. Since each element of L is the join of a subset of $J(L)$, L has at most 2^m elements. Strengthening a former result of Gao and Yu [10], it is shown in [6] that L satisfies Frankl's conjecture provided $|L| \geq 2^m - 2^{m/2}$. In the semimodular case we can prove more. For simplicity, finite lattices L with more than $5 \cdot 2^{m-3} = 2^m - \frac{3}{8} \cdot 2^m$ elements will be called *large.* The *height* $h(x)$ of an element $x \in L$ is the length (number of elements minus one) of any maximal chain in the principal ideal $\downarrow x$. (This makes sense, for any two maximal chains has the same length by semimodularity.)

Theorem 1 *Let L be a finite semimodular lattice. If L is large in the sense $|L| > 5 \cdot 2^{m-3}$, where $m = |J(L)|$, then L satisfies Frankl's conjecture.*

Proof Let $A(L)$ denote the set of atoms of L.

First we show that $|J(L) \setminus A(L)| \leq 1$. By way of contradiction, assume that a_1 and a_2 are distinct elements of $J(L) \setminus A(L)$. Let $a_3, \dots, a_m$ be the rest of nonzero join-irreducible elements, i.e., $J(L) = \{a_1, a_2, \dots, a_m\}$. Let B_m be the boolean lattice with atoms $x_1, \dots, x_m$, and consider both $B_m = (B_m; \vee, 0)$ and

$L = (L; \vee, 0)$ as join-semilattices with 0. Since B_m is the free join-semilattice with 0, there is a surjective homomorphism $\varphi\colon B_m \to L$, $x_i \mapsto a_i$. Let Θ denote the kernel of φ. Then, for $i = 1, 2$, the Θ-class $[x_i]$ of x_i is not a singleton, for otherwise a_i would be an atom. Since $a_i \neq 0$, we conclude that $0 \notin [x_i]$. Since Θ-classes are convex subsemilattices, there are elements $y_1 \in [x_1]$ and $y_2 \in [x_2]$ such that $y_1 \succ x_1$ and $y_2 \succ x_2$. They are distinct, for $a_1 \neq a_2$. Let $z = y_1 \wedge y_2$; it is an atom or the zero of B_m.

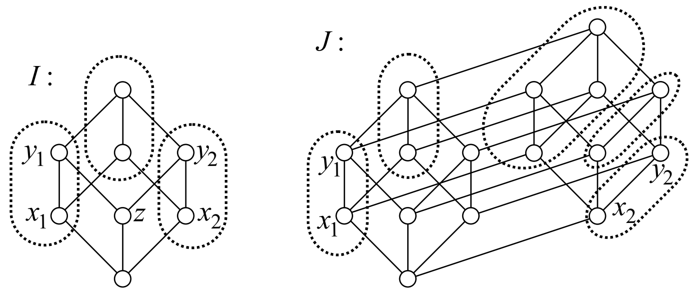

Fig. 1: Two ideals in B_m

First assume that z is an atom, and consider the ideal $I = \downarrow(y_1 \vee y_2)$ in B_m, cf. Figure 1. Let K denote the subsemilattice generated by those atoms of B_m that are not in I; K is not indicated in the figure. It follows from $(x_1, y_1), (x_2, y_2) \in \Theta$ that the restriction $\Theta|_I$ to I includes the semilattice congruence indicated in the figure. Hence Θ collapses I to five or less elements. For $u \in K$, let $u \vee I = \{u \vee t\colon t \in I\}$. If $(t_1, t_2) \in \Theta|_I$ then $(u \vee t_1, u \vee t_2) \in \Theta$. Hence Θ collapses $u \vee I$ to five or less elements. Now B_m is the union of the pairwise disjoint subsets $u \vee I$, $u \in B_m$. Therefore $L \cong B_m/\Theta$ consists of at most $5 \cdot |K| = 5 \cdot 2^{m-3}$ elements, which contradicts the assumption that L is large.

Secondly, assume that $z = 0$, and consider the ideal $J = \downarrow(y_1 \vee y_2)$, cf. Figure 1. Then the same argument as above gives $|L| \leq 9 \cdot 2^{m-4} < 5 \cdot 2^{m-3}$, a contradiction again. This proves that $|J(L) \setminus A(L)| \leq 1$.

Now, let us recall a well-known fact on semimodular lattices. An n-element subset $U = \{c_1, \dots, c_n\}$ of $A(L)$ is called *independent* if the sublattice $[U]$ generated by U is boolean with $A([U]) = U$. It is well-known, cf. e.g., Theorem IV.2.4 in Grätzer [11], that U is independent if and only if

$$(c_1 \vee \cdots \vee c_i) \wedge c_{i+1} = 0 \ \text{ for } \ i = 1, 2, \dots, n-1. \tag{1}$$

We need another, much easier version of independence: $U \subseteq J(L)$ will be called an *irredundant set* if $u \not\leq \bigvee(U \setminus \{u\})$ for every $u \in U$. In other words, $U = \{c_1, \dots, c_n\}$ is independent if no joinand can be omitted from $c_1 \vee \cdots \vee c_n$.

Now, armed with $|J(L) \setminus A(L)| \leq 1$, let us introduce some new notations. If $|J(L) \setminus A(L)| = 1$, then let a_1 be the only element of $J(L) \setminus A(L)$, let $a_2, \dots, a_k$ be the atoms in $\downarrow a_1$, and let $b_1, \dots, b_{m-k}$ be the rest of atoms. (Note that $k \geq 2$.) Otherwise, when $J(L) = A(L)$, let $k = 1$, let a_1 be an arbitrarily fixed atom, and let $b_1, \dots, b_{m-1}$ be the rest of atoms.

We claim that $|{\uparrow}a_1| \le |L|/2$. It suffices to show that for each $x \in {\uparrow}a_1$ there exists an $y = y(x) \in L \setminus {\uparrow}a_1$ such that $a \vee y = x$. (If there are several elements y with this property then we choose one of them.) Indeed, then the existence of the *injective* mapping ${\uparrow}a_1 \to L \setminus {\uparrow}a_1$, $x \mapsto y(x)$ will complete the proof. So, let $x \in {\uparrow}a_1$ be an arbitrary element. Then, clearly, there is an irredundant subset U of $J(L)$ whose join is x.

First let us assume that a_i is in U for some $1 \le i \le k$. Now we define $y = \bigvee\big(U \setminus \{a_i\}\big)$. Then $x = a_i \vee y$ and $a_i \le a_1 \le x$ gives $x = a_1 \vee y$ while the irredundance of U yields $a_i \not\le y$, implying $y \notin {\uparrow}a_1$.

Secondly, we assume that no a_i belongs to U. Then U is a set of atoms, say $U = \{b_1, \dots, b_n\}$. Using condition (1) and the irredundance of U we conclude that U is an independent set. Define $d_i = b_1 \vee \cdots \vee b_{i-1} \vee b_{i+1} \vee \cdots b_n$. Then the d_i, $1 \le i \le n$, are the coatoms of the boolean sublattice generated by U. If $a_1 \le d_i$ for all i, then $a_1 \le \bigwedge_{i=1}^{n} d_i = 0$, a contradiction. Hence we can select an $i \in \{1, \dots, n\}$ such that $a_1 \not\le d_i$. Then $y = d_i$ does the job, for $d_i = 0 \vee d_i \prec b_i \vee d_i = x$ by semimodularity, and $d_i < a_1 \vee d_i \le x$. □

Let us recall that finite, atomistic, semimodular lattices are *geometric lattices* by definition. Using the ideas around Figure 1, it is easy to see that $(x_1, y_1) \in \Theta$ implies that at leat 2^{m-2} elements of B_m are collapsed, i.e., L has at most $2^m - 2^{m-2} = 6 \cdot 2^{m-3}$ elements. This means that $|L| > 6 \cdot 2^{m-3}$ implies $J(L) = A(L)$ and $|[x_i]| = 1$ $(i = 1, \dots, m)$, whence the above proof clearly yields the following

Corollary 1 *Let L be a finite semimodular lattice with $|L| > 6 \cdot 2^{m-3}$, where $m = |J(L)|$. Then L is a geometric lattice, and for each atom f of L, $|{\uparrow}f| \le |L|/2$.*

If L has a Hasse diagram whose edges cross only at vertices then L is called a *planar lattice*. Recently, Grätzer and Knapp [12] has given a useful structure theorem for finite planar semimodular lattices; this is what the present paper relies on. Although this structure theorem is now generalized to all finite semimodular lattices in [7], we have been able to treat the planar case only.

If $a \parallel b$, then $S = \{a, b, a \wedge b, a \vee b\} \subseteq L$ will be called a *square* of L. If, in addition, $a \wedge b \prec a$ and $a \wedge b \prec b$, then S is called a *covering square*. By semimodularity, $a \vee b$ covers both a and b when S is a covering square. If each covering square of L is an interval then L is said to be *slim*. A mapping is called cover-preserving if it preserves the relation $\preceq$. Let us recall

Lemma 1 *(Grätzer and Knapp [12])*

- *Each finite planar slim semimodular lattice is a cover-preserving join-homomorphic image of the direct product of two finite chains.*
- *Each finite planar semimodular lattice can be obtained from a slim planar semimodular lattice by inserting new, doubly irreducible elements into some of its covering squares.*

Using the connection between Frankl's original conjecture and its lattice theoretic version, Roberts [16] yields that lattices with at most forty elements satisfy Frankl's conjecture. However, to explain why $|L| \geq 4$ is assumed in our main result below, we need only the obvious observation that lattices with less than four elements satisfy Frankl's conjecture.

Theorem 2 *Let L be a finite planar semimodular lattice consisting of at least four elements. Then L satisfies Frankl's conjecture. Moreover, at least one of the following two properties hold:*

- *either $1 \in J(L)$, and therefore $|{\uparrow} f| \leq |L|/4$ for $f = 1$,*
- *or there exist two distinct elements f_1 and f_2 in $J(L)$ such that $|{\uparrow} f_i| \leq |L|/2$ for $i = 1, 2$.*

Proof Let L be a finite planar semimodular lattice with $|L| \geq 4$. We will assume that L is not a chain and $1 \notin J(L)$, for otherwise the statement is evident.

First we consider the case when L is slim. We will treat it as a join-semilattice $(L, \vee)$. In virtue of Lemma 1, there are two chains, $\{0 < 1 < \cdots < n\}$ and $\{0 < 1 < \cdots < m\}$, and a join-congruence Θ of

$$D = \{0 < 1 < \cdots < n\} \times \{0 < 1 < \cdots < m\}$$

such that, up to isomorphism, $L = (L, \vee)$ equals D/Θ. (We will not use the cover-preserving property of the canonical $L \to L/\Theta$ homomorphism.) Since L is not a chain, $n \geq 2$ and $m \geq 2$. We assume that n and m are chosen such that $m + n$ is minimal, and we prove the slim case via induction on $m + n$. The smallest case, $m = n = 2$ is evident. So we assume that $m + n > 4$. For brevity, let $u = (n, 0)$, $v = (0, m)$, $1 = (m, n)$, $h = (n - 1, m)$, cf. Figure 2.

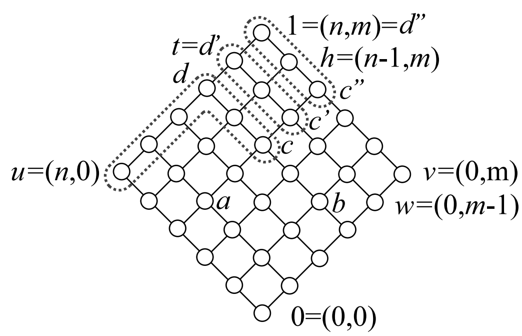

Fig. 2

Now we claim that $[u]\Theta$ and $[v]\Theta$ belong to $J(L) = J(D/\Theta)$. Their role is symmetric, so it suffices to deal with $[u]\Theta$. Suppose, by way of contradiction, that $[u]\Theta$ is not join-irreducible. Then there are $a, b \in D$ such that $[u]\Theta = [a]\Theta \vee [b]\Theta = [a \vee b]\Theta$ but $[a]\Theta < [u]\Theta$ and $[b]\Theta < [u]\Theta$, cf. Figure 2. (Although Figure 2 does not reflect the full generality, Θ is at least as large as indicated by dotted lines.) Let $c = a \vee b$. Since $[a]\Theta < [u]\Theta$ and $[b]\Theta < [u]\Theta$, we conclude that $a, b, c \in {\downarrow}h$. Let $c \prec c' \prec c'' \prec \cdots$ denote the unique maximal chain in the interval $[c, c \vee v] \subseteq {\downarrow}h$, and let $d = u \vee c$, $d' = u \vee c'$, $d'' = u \vee c'', \dots$ be the corresponding chain in the interval $[d, 1]$. Now, computing modulo Θ, for $x \in [u, d]$ we have $x = u \vee x \equiv c \vee x = d = u \vee c \equiv c \vee c = c$. Further, $d' = d \vee c' \equiv c \vee c' = c'$, $d'' = d \vee c'' \equiv c \vee c'' = c''$, etc. This means that each element of $[u, 1]$ is congruent to some element in ${\downarrow}h$ modulo Θ. Therefore, by the Third Isomorphism Theorem (cf. e.g., Thm. 6.18 in Burris and Sankappanavar [5]), $(L, \vee)$ is isomorphic to $({\downarrow}h)/\Psi$ where Ψ is the restriction of Θ to ${\downarrow}h$. However, this contradicts the minimality of $m + n$. We have seen that $[u]\Theta$ and $[v]\Theta$ are join-irreducible. $[u]\Theta = [0]\Theta$ is impossible, for otherwise L would clearly be a chain Finally, $[u]\Theta$ and $[v]\Theta$ are distinct, for otherwise $[u]\Theta = [u]\Theta \vee [v]\Theta = [u \vee v]\Theta = [1]\Theta$, a contradiction.

Now, we claim that

$$\big((0, i-1), (0, i)\big) \notin \Theta \ \text{ for } \ i = 1, \dots m. \tag{2}$$

By way of contradiction, suppose the opposite for some fixed i. Let Φ be the semilattice congruence of D whose two-element blocks are the $\{(j, i-1), (j, i)\}$, $j = 0, 1, \dots, n$, and all other blocks are singletons. Since

$$\big((j, i-1), (j, i)\big) = \big((j, 0) \vee (0, i-1), (j, 0) \vee (0, i)\big) \in \Theta,$$

we have $\Phi \subseteq \Theta$. Hence the Second Isomorphism Theorem (cf. e.g., Thm. 6.15 in [5]) gives that $(L, \vee)$ is a homomorphic image of $\{0 < 1 < \cdots < n\} \times \{0 < 1 < \cdots < m-1\}$, which contradicts the minimality of $m + n$.

Now, it follows from (2) that

$$|{\downarrow}[v]\Theta| \geq m + 1. \tag{3}$$

If, for $a \in D$, $[u]\Theta \leq [a]\Theta$ then $[a]\Theta = [u \vee a]\Theta$. This implies that

$$|{\uparrow}[u]\Theta| \leq m + 1. \tag{4}$$

We claim that

$${\uparrow}[u]\Theta \ \text{ is disjoint from } \ {\downarrow}[v]\Theta. \tag{5}$$

This comes easily, for in the opposite case we would have

$$[1]\Theta = [u \vee v]\Theta = [u]\Theta \vee [v]\Theta = [v]\Theta \in J(D/\Theta) = J(L),$$

which has been excluded previously. Now (3), (4) and (5) settle the slim case.

Finally, according to Lemma 1, the general case is obtained from the slim case via inserting new doubly irreducible elements into the interior (understood in geometrical sense in the Hasse diagram) of covering squares. Since ${\uparrow}[u]\Theta$ and ${\uparrow}[v]\Theta$ are chains, they include no covering square. Hence no new element is inserted into them. I.e., the size of ${\uparrow}[u]\Theta$ and that of ${\uparrow}[v]\Theta$ remain fixed while the size of L increases. $\square$

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