Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **47** (2008) 55–75

Furi–Pera Fixed Point Theorems in Banach Algebras with Applications

Smaïl DJEBALI [1], Karima HAMMACHE [2]

Department of Mathematics, E.N.S.,
Po Box 92, 16050 Kouba, Algeria
[1] *e-mail: djebali@ens-kouba.dz (corresponding author)*
[2] *e-mail: k.hammache@hotmail.com*

(Received February 17, 2008)

Abstract

In this work, we establish new Furi–Pera type fixed point theorems for the sum and the product of abstract nonlinear operators in Banach algebras; one of the operators is completely continuous and the other one is $\mathcal{D}$-Lipchitzian. The Kuratowski measure of noncompactness is used together with recent fixed point principles. Applications to solving nonlinear functional integral equations are given. Our results complement and improve recent ones in [10], [11], [17].

Key words: Banach algebra; Furi–Pera condition; fixed point theorem; measure of noncompactness; integral equations.

2000 Mathematics Subject Classification: 47H10,
secondary: 47H09, 47H30, 47J25, 46J10

1 Introduction

In many areas of natural sciences, mathematical physics, mechanics and population dynamics, problems are modeled by mathematical equations which may be reduced to perturbed nonlinear equations of the form:

$$Ay + By = y, \quad y \in M$$

where M is a closed, convex subset of a Banach space X, and A, B are two nonlinear operators. A useful tool to deal with such problems is the celebrated fixed point theorem due to Krasnozels'kĭi, 1958 (see [15, 16]):

Theorem 1.1 *Let M be a nonempty closed convex subset of a Banach space X and A, B be two maps from M to X such that*
(a) A is compact and continuous,
(b) B is a contraction,
(c) $Ax + By \in M$, for all $x, y \in M$.
Then $A + B$ has at least one fixed point in M.

We recall the

Definition 1.1 Let E be a Banach space and $f\colon E \to E$ be a mapping. Then f is said compact if $f(E)$ is compact. It is called totally bounded whenever $f(A)$ is relatively compact for any bounded subset A of E. Finally, f is completely continuous if is continuous and totally bounded.

The proof of Theorem 1.1 combines the metric Banach contraction mapping principle both with the topological Schauder's fixed point theorem [1, 7, 17, 19] and uses the fact that if E is a linear vector space, $F \subset E$ a nonempty subset and $g\colon F \to E$ a contraction, then the mapping $I - g\colon F \to (I - g)(F)$ is an homeomorphism.

In 1998, Burton [6] noticed that the Krasnozels'kĭi fixed point theorem remains valid if condition (c) is replaced by the following less restrictive one:

$$\forall y \in M,\ (x = Ay + Bx) \Longrightarrow x \in M. \tag{1}$$

However, the study of some integral equations involving the product of operators rather than the sum may be considered only in the framework of Banach algebras for which Dhage proved in 1988 the following

Theorem 1.2 [9] *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A, B\colon S \to S$ be two operators such that*
(a) A is Lipschitzian with a Lipschitz constant α.
(b) $\left(\frac{I}{A}\right)^{-1}$ exists on $B(S)$, where I is an identity operator and the operator

$$\frac{I}{A}\colon X \to X \quad \textit{is defined by} \quad \left(\frac{I}{A}\right)(x) = \frac{x}{Ax}.$$

(c) B is completely continuous.
(d) $AxBy \in S$, for all $x, y \in S$.
Then the operator equation $x = AxBx$ has a solution, whenever $\alpha M < 1$, where $M := \|B(S)\| = \sup\{\|Bx\|\colon x \in S\}$.

Remark 1.1 Note that $\left(\frac{I}{A}\right)^{-1}$ exists if the operator $\frac{I}{A}$ is well defined and is one-to-one. In [10], the author improved Theorem 1.2 by removing the restrictive condition (b). The proof of the improved theorem involves the measure of noncompactness theory (see Section 2). Also, the assumption stating that A is Lipschitzian is extended to $\mathcal{D}$-Lipschitzian mappings according to the following definition. Finally, Assumption (d) is weakened to Burton's relaxed condition.

Definition 1.2 Let E be a Banach space and $f\colon E \to E$ a mapping.

(a) f is called $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ_f if there exists a continuous nondecreasing function $\phi_f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi_f(0) = 0$ and

$$\|f(x) - f(y)\| \leq \phi_f(\|x - y\|),\ \forall (x, y) \in E^2.$$

(b) Moreover if $\phi_f(r) < r$, $\forall r > 0$, then f is called nonlinear contraction.

(c) In particular, if $\phi_f(r) = kr$ for some constant $0 < k < 1$, then f is a contraction.

(d) f is said non-expansive if $\phi_f(r) = r$, that is

$$\|f(x) - f(y)\| \leq \|x - y\|,\ \forall (x, y) \in E^2.$$

Immediately, we have

Lemma 1.1 *Every $\mathcal{D}$-Lipschitzian mapping A is bounded, i.e. maps bounded sets into bounded sets.*

Proof Let S be a bounded subset in a Banach space E and $d = \operatorname{diam} S$ where $\operatorname{diam} S$ stands for the diameter of S. Let $s_0 \in S$ be fixed. Since ϕ_A is nondecreasing, for any $s \in S$, we have

$$\|As\| \leq \|As_0\| + \|As - As_0\| \leq \|As_0\| + \phi_A(\|s - s_0\|) \leq \|As_0\| + \phi_A(d),$$

whence comes the result. □

Next, we state three basic existence results, important for the rest of the paper:

Theorem 1.3 ([10, Thm 2.1, p. 275]) *Let S be a closed, convex and bounded subset of a Banach algebra E and let $A, B\colon S \longrightarrow E$ be two operators such that*
(a) A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ_A.
(b) B is completely continuous.
(c) ($x = AxBy \Rightarrow x \in S$), for all $y \in S$.
Then the operator equation $x = AxBx$ has a solution, whenever $\phi_A(r) < r$, $\forall r > 0$ where $M := \|B(S)\|$.

The idea of extending contractions to nonlinear contractions comes from Boyd and Wong fixed point theorem which we recall hereafter for completeness; this theorem generalizes the Banach fixed point principle, dating 1922 (see e.g., [19]).

Theorem 1.4 ([5], 1969) *Let E be a Banach space and $f : E \to E$ a nonlinear contraction. Then f has a unique fixed point in E.*

In practice, condition (c) in Theorem 1.3 is not so easy to come by as it is the case in Schauder's fixed point theorem where a compact mapping is asked to map a ball into itself. In 1987, Furi and Pera introduced a new condition instead and proved the following fixed point theorem in the general framework of Fréchet spaces:

Theorem 1.5 (see [14] or [1, Thm 8.5, p. 99]) *Let E be a Fréchet space, Q a closed convex subset of E, $0 \in Q$ and let $T\colon Q \to E$ be a continuous compact mapping. Assume further that*

$$(\mathcal{FP}) \qquad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j\geq 1} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x,\lambda) \text{ with } x = \lambda F(x) \text{ and } 0 \leq \lambda < 1, \\ \text{then } \lambda_j F(x_j) \in Q \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then T has a fixed point in Q.

Our aim in this paper is to prove new existence theorems of Dhage type with the condition (c) in Theorem 1.3 replaced by the Furi–Pera condition $(\mathcal{FP})$. More precisely, we will consider mappings of the form $F = AB + C$ where B is completely continuous and A, C are $\mathcal{D}$-Lipchitzian while F satisfies the Furi–Pera condition. This is the content of Theorems 3.1 and 3.2. In Theorem 3.3, the Furi–Pera condition is verified by another mapping, denoted N. The latter is proved to fulfill Boyd and Wong fixed point theorem (Theorem 1.4). As a consequence, we derive some known fixed point theorems obtained recently in [10, 11, 17]. The proofs are detailed in Sections 4 and 5. A further result where we relax condition (c) in Theorem 1.3 is given in Section 6. Some applications to functional nonlinear integral equations are provided in Section 7. We end the paper with some concluding remarks in Section 8. The notation := means throughout to be defined equal to. $\mathcal{B}_r(x)$ will denote the open ball in a metric space X, centered at x and of radius r and $\mathbb{R}^+$ will refer to the set of all positive real numbers. Before we present the main results of this paper, some auxiliary results are recalled hereafter.

2 Preliminaries

Definition 2.1 Let E be a Banach space and $\mathsf{B} \subset \mathcal{P}(E)$ the set of bounded subsets of E. For any subset $A \in \mathsf{B}$, define $\alpha(A) = \inf D$ where

$$D = \{\varepsilon > 0\colon A \subset \cup_{i=1}^{i=n} A_i,\ \operatorname{diam}(A_i) \leq \varepsilon,\ \forall i = 1, \ldots, n\}.$$

α is called the Kuratowski measure of noncompactness, $\alpha - MNC$ for short. Hereafter, we gather together its main properties. For more details, we refer to [3, 4, 7].

Proposition 2.1 *For any $A, B \in \mathsf{B}$, we have*
(a) $0 \leq \alpha(A) \leq \operatorname{diam}(A)$
(b) $A \subseteq B \Rightarrow \alpha(A) \leq \alpha(B)$ *(α is nondecreasing).*
(c) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$.
(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ *(α is lower-additive).*
(f) $\alpha(\operatorname{Conv} A)) = \alpha(\overline{A}) = \alpha(A)$.
(h) $\alpha(A) = 0 \Rightarrow A$ *is relatively compact.*

Definition 2.2 Let E_1, E_2 be two Banach spaces and $f : E_1 \to E_2$ a continuous application which maps bounded subsets of E_1 into bounded subsets of E_2
(a) f is called α-Lipschitz if there exists some $k \geq 0$ such that

$$\alpha(f(A)) \leq k\alpha(A),$$

for any bounded subset $A \subset E_1$.
(b) f is a strict α-contraction when $k < 1$.
(c) f is said to be α-condensing whenever

$$\alpha(f(A)) < \alpha(A),$$

for any bounded subset $A \subset E_1$ with $\alpha(A) \neq 0$.

Remark 2.1 Clearly, the case $k = 0$ corresponds to f totally bounded which is of course α-condensing.

To develop further arguments, we need the following auxiliary results. They extend Theorem 1.5 to α-condensing and α-Lipschitz maps in Banach spaces, respectively (for the proofs, we refer to [17]).

Theorem 2.1 *Let E be a Banach space and Q a closed convex bounded subset of E with $0 \in Q$. In addition, assume $F\colon Q \to E$ is an α-condensing map which satisfies the Furi–Pera condition. Then F has a fixed point $x \in Q$.*

Theorem 2.2 *Let E be a Banach space and Q a closed convex bounded subset of E with $0 \in Q$. In addition, assume that $(I - F)(S)$ is a closed, $F\colon Q \to E$ is an α-Lipschitz map with $k = 1$ and satisfies the Furi–Pera condition. Then F has a fixed point $x \in Q$.*

3 Main results

We state the following main theorems of this paper.

Theorem 3.1 *Let S be a closed, convex and bounded subset of a Banach algebra X with $0 \in S$ and let $A, C\colon X \to X$ and $B\colon S \to X$ be three operators such that*

(a) A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively.

(b) B is completely continuous.

(c) The operator $F\colon S \to X$ defined by

$$F(x) = AxBx + Cx$$

satisfies the Furi–Pera condition.
Then the abstract equation $x = AxBx + Cx$ has a solution $x \in S$ whenever

$$(\mathcal{H}_0) \qquad M\phi_A(r) + \phi_C(r) < r, \quad \forall r > 0$$

where $M = \|B(S)\|$.

Remark 3.1 More generally, one may consider n $\mathcal{D}$-Lipschitzian operators A_i ($i = 1, \cdots, n$) with $\mathcal{D}$-functions ϕ_i and completely continuous operators B_i ($i = 1, \cdots, n$) defined on a closed, bounded, convex subset S of a Banach algebra containing 0 and satisfying

$$\sum_{i=1}^{n} M_i \phi_i(r) < r$$

where, for each i, $M_i = \|B_i(S)\|$. If the operator

$$F(x) = \sum_{i=1}^{n} (A_i B_i)(x)$$

satisfies the Furi–Pera condition, then the abstract nonlinear equation

$$\sum_{i=1}^{n} (A_i B_i)(x) = x$$

has a solution $x \in S$.

Remark 3.2 Let $S = \mathcal{B}_R(0)$. It is well known that $F(\partial S) \subset S$ implies the Furi–Pera condition. Assume further that $A0 = C0 = 0$. Then also Assumption $(\mathcal{H}_0)$ implies the Furi–Pera condition. Indeed, let $(x_j, \lambda_j)_{j \geq 1}$ be a sequence in $\partial S \times [0, 1]$ converging to some limit (x, λ) with $x = \lambda F(x)$ and $0 \leq \lambda < 1$, then

$$\|\lambda_j F x_j\| \leq \|Ax_j\| \|Bx_j\| + \|Cx_j\| \leq M\phi_A(\|x_j\|) + \phi_C(\|x_j\|) \leq \|x_j\| = R.$$

Hence $\lambda_j F(x_j) \in S$, $\forall j \in \mathbb{N}^*$.

Theorem 3.2 *Let S be a closed, convex and bounded subset of a Banach algebra X with $0 \in S$ and let $A, C\colon X \to X$ and $B\colon S \to X$ be three operators such that*

(a) A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively.

(b) B is completely continuous.

(c) The operator $F\colon S \to X$ defined by $F(x) = AxBx + Cx$ satisfies the Furi–Pera condition.

Then the abstract equation $x = AxBx + Cx$ has a solution $x \in S$ provided $(I - F)(S)$ is closed and the large inequality $M\phi_A(r) + \phi_C(r) \leq r$, $\forall r > 0$ holds.

Theorem 3.3 *Let S be a closed, convex and bounded subset of a Banach algebra X with $0 \in S$ and let $A\colon X \to X$ and $B\colon S \to X$ be two operators such that*

(a) A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ_A.

(b) B is completely continuous.

(c) The operator $N\colon S \to X$ defined by $Nx = y$, where y is the unique solution of the operator equation $y = AyBx$, satisfies the Furi–Pera condition.

Then the operator equation $x = AxBx$ has a solution $x \in S$ provided

$$(\mathcal{H}) \qquad \begin{cases} \text{the mapping } \Phi\colon [0, +\infty) \to [0, +\infty) \\ r \mapsto \Phi(r) = r - M\phi_A(r) \text{ is increasing to infinity.} \end{cases}$$

Remark 3.3 (a) Obviously, Assumption $(\mathcal{H})$ implies that for any $r > 0$ $M\phi_A(r) < r$ and amounts to the restriction $0 < k < \frac{1}{M}$ in case ϕ_A is a k-contraction.

(b) The condition (c) in Theorem 3.3 is different from the condition (c) in Theorem 3.1 which means that $N(S) \subset S$. One can make analogy with the condition (c) in Theorem 1.1 compared with Burton's weak condition (1). Regarding these conditions, a discussion is given in the concluding remarks (see Section 7).

3.1 Some consequences

In this section, we derive four existence principles. In particular, we recover some known results: the first one is Dhage's Theorem 1.3 ([10, Thm 2.1]); the second one is a nonlinear alternative in Banach algebras ([11, Thm 2.2]), the third one is concerned with the case of the sum of a nonlinear contraction and a compact mapping ([17, Thm 2.2]) while the last one is a useful classical result.

Corollary 3.1 *Assume Assumptions (a)–(c) in Theorem 1.3 are satisfied,* $0 \in S$ *where* S *is either a ball or any subset homeomorphic to a bounded, closed convex subset and* $AB(S) \subset S$*. Then, the same conclusion of this theorem holds true provided* $(\mathcal{H})$ *is fulfilled.*

Proof We prove the corollary first in case $S = \mathcal{B}_M(0)$ and then when AB maps S into itself. In the latter case, S could be any subset homeomorphic to a closed, bounded and convex subset of X.

Step 1: $S = \mathcal{B}_M(0)$. We only have to check that condition (c) in Theorem 1.3 implies the Furi–Pera condition (c) in Theorem 3.3. For this, let $(x_j, \lambda_j)_{j\geq 1}$ be a sequence in $\partial S \times [0,1]$ converging to some limit (x, λ) with $x = \lambda N x$, $0 \leq \lambda < 1$ and show that $\lambda_j N(x_j) \in S$ for j large enough. For any $j \in \mathbb{N}^*$, $\|\lambda_j N(x_j)\| \leq \|N(x_j)\| = \|y_j\|$ where $y_j = Ay_jBx_j$. Since $x_j \in \partial S \subset S$ and condition (c) of Theorem 1.3 is satisfied, $y_j \in S$. Hence $\|y_j\| \leq M$. This implies that $\|\lambda_j N(x_j)\| \leq M$. Our claim, that is $\lambda_j N(x_j) \in S$, is then proved.

Step 2: $AB(S) \subset S$. By the Dugundji's extension theorem (see [7, 19]), let $r\colon X \to S$ be a retraction and let $\mathcal{B}$ be a ball containing S. Then consider the diagram

$$\mathcal{B} \xrightarrow{r} S \xrightarrow{AB} S.$$

From Step 1, the map $AB \circ r$ has a fixed point $x \in \mathcal{B}$, that is satisfying $Ar(x)Br(x) = x$. Since $ABr(x) \in S$, it follows that $x \in S$ and thus $r(x) = x = ABx$.

Step 3: S is homeomorphic to $\tilde{S}$, where $\tilde{S}$ is a closed, bounded and convex subset of X. Let the diagram

$$\tilde{S} \xrightarrow{h^{-1}} S \xrightarrow{AB} S \xrightarrow{h} \tilde{S},$$

where h is an homeomorphism. From Step 2, there exists some $y \in \tilde{S}$ such that $h \circ AB \circ h^{-1}(y) = y$. Then $ABx = x$, for $x = h^{-1}(y) \in S$, ending the proof of the corollary. □

Corollary 3.2 ([11, Thm 2.2, p. 272]) *Let X be a Banach algebra and let $A, B, C\colon X \longrightarrow X$ be three operators satisfying $(\mathcal{H}_0)$ together with*
(a) A and C are $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-functions ϕ_A and ϕ_C respectively.
(b) B is completely continuous.
Then
(a) either $F = AB + C$ has a fixed point in X,
(b) or the set $\{x \in X,\ \lambda F(x) = x,\ 0 < \lambda < 1\}$ is unbounded.

Proof Assume Alternative (b) does not hold true. Then, there exists some positive constant R such that

$$\forall \lambda \in (0,1),\ (\lambda F(x) = x \Rightarrow \|x\| \leq R). \tag{2}$$

In order to show that F satisfies the Furi–Pera condition, consider a sequence $(x_j, \lambda_j)_{j\geq 1} \in \partial S \times [0,1]$ converging to some limit (x, λ) with $x = \lambda F(x)$ and $0 < \lambda < 1$, where $S = \overline{\mathcal{B}}_{R+1}(0)$. By continuity of F, we have that

$$\|\lambda_j F(x_j)\| \leq \|\lambda F(x)\| + 1, \text{ for sufficientlly large } j. \tag{3}$$

Since $x = \lambda F(x)$, (2) yields

$$\|\lambda F(x)\| = \|x\| \leq R.$$

This with (3) imply that $\lambda_j F_j(x) \in S$. Our claim, namely Alternative (a), then follows from Theorem 3.1. □

The following two particular cases of Theorem 3.1 are useful in practice.

Corollary 3.3 ([17, Thm 2.2, p. 3]) *Let S be a closed, convex and bounded subset of a Banach space X with $0 \in S$ and let $F_1\colon X \to X$ and $F_2\colon S \to X$ be two operators such that*
(a) F_1 is a nonlinear contraction.
(b) F_2 is completely continuous.
(c) The sum $F = F_1 + F_2\colon S \to X$ satisfies the Furi–Pera condition $(\mathcal{FP})$.
Then F has a fixed point $x \in S$.

Proof Take $B = F_2,\ C = F_1,\ A \equiv 1$ and then $\varphi_A \equiv 0$. □

Corollary 3.4 *Let S be a closed, convex and bounded subset of a Banach algebra X with $0 \in S$ and let $A, C\colon X \to X$ and $B\colon S \to X$ be three operators such that*
(a) A and C are Lipschitzian with Lipschitz constants k_A and k_C respectively.
(b) B is completely continuous.
(c) The operator $F\colon S \to X$ defined by $F(x) = AxBx + Cx$, $x \in X$ satisfies the Furi–Pera condition $(\mathcal{FP})$.
Then the equation $x = AxBx + Cx$ has a solution $x \in S$ whenever $k_A\|B(S)\| + k_C < 1$.

4 Proofs of Theorems 3.1 and 3.2

The proofs are direct consequences of the following lemma:

Lemma 4.1 *Under Assumptions (a), (b) of Theorem 3.1 together with $(\mathcal{H}_0)$, the map $F\colon S \to X$ defined by $F(x) = AxBx + Cx$ is α-condensing.*

Proof Let $D \subset S$ be a bounded subset and $\delta > 0$. There exists a covering $(D_i)_{i=1}^n$ such that $D \subset \bigcup_{i=1}^n D_i$ and $\mathrm{diam}(D_i) \le \alpha(D) + \delta$, for each $i = 1, \ldots, n$. For every $i \in \{1, \cdots, n\}$, let $x_1^i = x_1$, $x_2^i = x_2 \in D_i$ and $E_i = F(D_i)$. Clearly $F(D) \subset \bigcup_{i=1}^n E_i$. In addition, we have the estimates

$$\begin{aligned} &\|F(x_1) - F(x_2)\| = \|Ax_1Bx_1 + Cx_1 - Ax_2Bx_2 - Cx_2\| \\ &\le \|Ax_1\|\,\|Bx_1 - Bx_2\| + \|Bx_2\|\,\|Ax_1 - Ax_2\| + \|Cx_1 - Cx_2\| \\ &\le \|Ax_1\|\,\mathrm{diam}\big(B(D_i)\big) + M\,\|Ax_1 - Ax_2\| + \|Cx_1 - Cx_2\| \\ &\le \|Ax_1\|\,\alpha\big(B(D_i)\big) + M\,\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

Since B is completely continuous, $\alpha\big(B(D_i)\big) = 0$, for each $i \in \{1, \cdots, n\}$ follows from Proposition 2.1(e). We infer that

$$\|F(x_1) - F(x_2)\| \le M\,\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|).$$

Since ϕ_A and ϕ_C are non decreasing, it follows that

$$\begin{aligned} \mathrm{diam}\,E_i &\le M\,\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|) \\ &\le M\phi_A(\mathrm{diam}(D_i)) + \phi_C(\mathrm{diam}(D_i)) \\ &\le M\phi_A(\alpha(D) + \delta) + \phi_C(\alpha(D) + \delta). \end{aligned}$$

Therefore

$$\alpha(F(D)) \le M\phi_A(\alpha(D) + \delta) + \phi_C(\alpha(D) + \delta).$$

Since $\delta > 0$ is arbitrary, we deduce the estimate

$$\alpha(F(D)) \le M\phi_A(\alpha(D)) + \phi_C(\alpha(D)).$$

Taking into account Assumption $(\mathcal{H}_0)$, we arrive at

$$\alpha\big(F(D)\big) < \alpha(D),$$

proving our claim. □

Remark 4.1 *It is well known (see e.g. [17, proof of Thm 2.1]) that the sum of a nonlinear contraction and a compact mapping is α-condensing. Lemma 4.1 extends this result to $\mathcal{D}$-Lipschitz mappings as well as to the product of a compact and a $\mathcal{D}$-Lipschitz maps.*

Proof of Theorem 3.1. By Lemma 4.1, the map $F\colon S \to X$ defined by $F(x) = AxBx + Cx$ is α-condensing. Since F satisfies the Furi–Pera condition, it follows by Theorem 2.1 that F has at least one fixed point $x \in S$, solution of the equation $x = AxBx + Cx$, ending the proof of the theorem. □

Proof of Theorem 3.2. Since the mapping $F\colon S \to X$ defined by $F(x) = AxBx + Cx$ is α-condensing by Lemma 4.1, then it is α-Lipschitz with $k = 1$. Moreover $(I-F)(S)$ is closed and F satisfies the Furi–Pera condition. Therefore, Theorem 2.2 implies that F has at least one fixed point $x \in S$, solution of the equation $x = AxBx + Cx$. □

5 Proof of Theorem 3.3

The proof follows from Theorem 2.1 once we have proved the following two technical lemmas.

Lemma 5.1 *Under the assumptions of Theorem 3.3, the operator $N\colon X \to X$ introduced in condition (c) is well defined and is bounded (on bounded subsets of X).*

Proof For any $x \in S$, let the mapping A_x be defined in X by $A_x y = AyBx$. Then, for any $y_1, y_2 \in X$,

$$\|A_x y_1 - A_x y_2\| = \|Bx\|\|Ay_1 - Ay_2\| \leq \|Bx\|\phi_A\left(\|y_1 - y_2\|\right) \leq M\phi_A\left(\|y_1 - y_2\|\right)$$

with $M\phi_A(r) < r$, $\forall r > 0$. By the Boyd and Wong fixed point theorem (see Theorem 1.4), A_x has only one fixed point $y \in X$ and so the mapping N is well defined. In addition, let $D \subset X$ be any bounded subset, $x \in D$ and $y = Nx$ where y is the unique solution of the equation $y = AyBx$. Thus

$$\|y\| = \|Bx\|\|Ay\| \leq M\|Ay\|.$$

Let $y_0 \in X$. With Assumption $(\mathcal{H})$, we have the following estimates

$$\|y\| \leq M\left(\|Ay - Ay_0\| + \|Ay_0\|\right) \leq M\phi_A\left(\|y - y_0\|\right) + M\|Ay_0\|.$$

Hence

$$\|y - y_0\| \leq \|y\| + \|y_0\| \leq M\phi_A\left(\|y - y_0\|\right) + M\|Ay_0\| + \|y_0\|.$$

It follows that

$$\Phi(\|y - y_0\|) = \|y - y_0\| - M\phi_A(\|y - y_0\|) \leq M\|Ay_0\| + \|y_0\|.$$

This in turn implies successively

$$\begin{aligned} &\|y - y_0\| \leq \Phi^{-1}(M\|Ay_0\| + \|y_0\|) \\ &\|y\| \leq \|y - y_0\| + \|y_0\| \leq \Phi^{-1}(M\|Ay_0\| + \|y_0\|) + \|y_0\|, \end{aligned}$$

proving our claim. □

Remark 5.1 To prove Lemma 5.1, we only need $\lim_{s\to+\infty}\Phi(s)=+\infty$ without the increasing character of Φ.

Lemma 5.2 Under the hypotheses of Theorem 3.3, the operator N introduced in the condition (c) is compact.

Proof

Claim 1. N is continuous. Let (x_n) be a sequence in S converging to some limit x. Since S is closed, $x\in S$. Moreover

$$\begin{aligned}\|Nx_n-Nx\| &= \|ANx_nBx-ANxBx\|\\ &\leq \|ANx_nBx_n-ANxBx_n\|+\|ANxBx_n-ANxBx\|\\ &\leq \|Bx_n\|\|ANx_n-ANx\|+\|ANx\|\|Bx_n-Bx\|\\ &\leq M\phi(\|x_n-x\|)+\|ANx\|\|Bx_n-Bx\|.\end{aligned}$$

Whence

$$\limsup_{n\to\infty}\|Nx_n-Nx\|\leq M\phi(\limsup_{n\to\infty}\|x_n-x\|)+\|ANx\|\limsup_{n\to\infty}\|Bx_n-Bx\|.$$

From Assumption (b), B is continuous; hence

$$\limsup_{n\to\infty}\|Nx_n-Nx\|=0,$$

yielding the continuity of N.

Claim 2. N is compact. From Lemmas 1.1 and 5.1, there exists some positive constant k_1 such that $\|ANx\|\leq k_1$, $\forall x\in S$. Let $\varepsilon>0$ be given. Since S is bounded and B is completely continuous, $B(S)$ is relatively compact. Then there exists a set $\mathcal{E}=\{x_1,\dots,x_n\}\subset S$ such that

$$B(S)\subset\bigcup_{i=1}^{n}\mathcal{B}_\delta(w_i)$$

where $w_i:=B(x_i)$ and $\delta:=k_2\varepsilon$ for some constant k_2 to be selected later on. Therefore, for any $x\in S$, there exists some $x_i\in\mathcal{E}$ such that

$$0\leq\|Bx-Bx_i\|\leq k_2\varepsilon.$$

We have

$$\begin{aligned}\|Nx_i-Nx\| &= \|ANx_iBx_i-ANxBx\|\\ &\leq \|ANx_iBx_i-ANxBx_i\|+\|ANxBx_i-ANxBx\|\\ &\leq \|Bx_i\|\|ANx_i-ANx\|+\|ANx\|\|Bx_i-Bx\|\\ &\leq M\phi_A(\|Nx_i-Nx\|)+k_1k_2\varepsilon. \qquad (4)\end{aligned}$$

Hence

$$\Phi(\|Nx_i-Nx\|)=\|Nx_i-Nx\|-M\phi_A(\|Nx_i-Nx\|)\leq k_1k_2\varepsilon.$$

From Assumption $(\mathcal{H})$, it follows that

$$\|Nx_i - Nx\| \leq \Phi^{-1}(k_1 k_2 \varepsilon).$$

Choosing $0 < k_2 \leq \frac{\Phi(\varepsilon)}{k_1 \varepsilon}$, we obtain

$$\|Nx_i - Nx\| \leq \varepsilon.$$

We have proved that $N(S) \subset \bigcup_{i=1}^{n} \mathcal{B}_\varepsilon(Nx_i)$, showing that N is totally bounded and ending the proof of Lemma 5.2. □

Remark 5.2 In Claim 2, we correct the proof of Theorem 2.1, p. 275 of [10] where ϕ_A was taken $\phi_A(r) \leq \alpha r$, $r > 0$. To this end, Assumption $(\mathcal{H})$ is essential.

Remark 5.3 In Theorem 3.3, the condition that S is unbounded may be relaxed when B completely continuous is replaced by B compact, that is $B(S)$ relatively compact. Indeed, the proof of Lemma 5.2 remains unchanged and then we rather apply Theorem 1.5.

6 A further result

In the following, we prove that the condition (c) in Theorem 1.3 may be relaxed.

Theorem 6.1 *Let S be a closed, convex and bounded subset of a Banach algebra E such that $\operatorname{int} S \neq \emptyset$ and let $A, B\colon S \to E$ be two operators such that*
(a) A is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ_A.
(b) B is completely continuous.
(c') $(x = AxBy \Rightarrow x \in S)$, for all $y \in \partial S$.
Then the operator equation $x = AxBx$ has a solution, whenever $M\phi_A(r) < r$, $\forall r > 0$.

Proof Let $r\colon X \to S$ be a retraction. Moreover using the Minkowski functional (see [18, Lemma 4.2.5, p. 27]), r may be chosen so that

$$\begin{cases} r(x) = x, & x \in S \\ r(x) \in \partial S, & x \notin S. \end{cases}$$

We claim that $Nr\colon X \to X$ has a fixed point (here we denote by $fg = f \circ g$). Since N is completely continuous by Lemma 5.2 and r is continuous, the composite $rN\colon S \to S$ is completely continuous. Then Schauder's fixed point theorem implies that rN has a fixed point, i.e. there exists some $x_0 \in S$ such that $rNx_0 = x_0$. As a consequence, Nr has a fixed point. Indeed, letting $y_0 = Nx_0$, we get

$$ry_0 = rNx_0 = x_0 \Longrightarrow Nry_0 = Nx_0 = y_0.$$

That is $y_0 = Ay_0Bry_0$. Since $ry_0 \in \partial S$, assumption (c') implies that $y_0 \in S$, ending the proof of the theorem. □

Remark 6.1 One may take the subset S unbounded and the operator B compact and then apply Rothe's Theorem to prove Theorem 6.1. This is the main motivation of the study of the topological structure of the subsubset $\mathcal{F}_{Nr} := \{x \in X,\, x = Nrx\} \subset S$ which is nonempty by Theorem 6.1.

(a) $\mathcal{F}_{Nr}$ *is closed.* Let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{F}_{Nr}$ be a sequence such that $x_n \to x$, as $n \to +\infty$. We show that $x \in \mathcal{F}_{Nr}$. First $x_n = Nr(x_n)$ and, since N and r are continuous, $\lim_{n\to+\infty} Nr(x_n) = Nr(x)$. Then, the uniqueness of the limit implies that $x = Nr(x)$ yielding $x \in \mathcal{F}_{Nr}$.

(b) $\mathcal{F}_{Nr}$ *is compact.* Indeed, $r(\mathcal{F}_{Nr}) \subset S$ implies $N(r(\mathcal{F}_{Nr})) \subset N(S)$ and then $\alpha(N(r(\mathcal{F}_{Nr}))) \leq \alpha(N(S)) = 0$ because N is compact by Lemma 5.2. Here α is the measure of noncompactness (see Section 2). In addition, $\mathcal{F}_{Nr} \subset Nr(\mathcal{F}_{Nr})$ implies that $\alpha(\mathcal{F}_{Nr}) \leq \alpha(Nr(\mathcal{F}_{Nr})) \leq 0$; then $\alpha(\mathcal{F}_{Nr}) = 0$ and our claim follows.

7 Applications

7.1 Example 1

Let $X = C([0,1],\mathbb{R})$ be the Banach Algebra of real continuous functions defined on the interval $[0,1]$ endowed with the sup-norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

For some sufficiently large positive real number R, let $S = \overline{\mathcal{B}}_R(0)$ be the closed ball centered at the origin and with radius R. Consider the nonlinear functional integral equation, for $t \in [0,1]$ and the parameter α lies in the interval $(0,1)$

$$x(t) = \left(1 + \frac{\alpha R}{R+1}|x(\mu(t))|\right)\left(q(\theta(t)) + \int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)\, ds\right) \tag{5}$$

where the functions $\mu, \theta, \sigma, \eta\colon [0,1] \to [0,1]$ are continuous. Assume that $q\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous functions and satisfy

$$|g\big(t,x)| \leq 1 - \|q\|_\infty, \quad \forall (t,x) \in [0,1]\times\mathbb{R}, \tag{6}$$

where $\|q\|_\infty := \max_{t\in[0,1]} |q(t)|$. Let the mappings A and B be defined by

$$A\colon X \to X, \qquad Ax(t) = 1 + \frac{\alpha R}{R+1}|x(\mu(t))|$$

and

$$B\colon S \to X, \qquad Bx(t) = q(\theta(t)) + \int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)\, ds.$$

Then the integral equation (5) is equivalent to the operator equation

$$Ax(t)Bx(t) = x(t), \quad t \in [0,1].$$

(a) *Properties of the mappings A, B.* Clearly, A is a Lipchitzian map with constant $k = \frac{R}{R+1}\cdot$ To prove B is completely continuous, let $(x_n)_{n\in\mathbb{N}} \subset X$. Since

$$|B(x_n)(t)| \leq |q(\theta(t))| + \left|\int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)\, ds\right| = \|q\| + (1 - \|q\|) = 1,$$

the sequence $(B(x_n))_{n\in\mathbb{N}}$ is uniformly bounded. Moreover B is equi-continuous. To see this, let $t_1, t_2 \in [0,1]$; then

$$\begin{aligned} |B(x_n)(t_1) - B(x_n)(t_2)| &\leq |q(\theta(t_1)) - q(\theta(t_2))| + \left|\int_{\sigma(t_1)}^{\sigma(t_2)} g\big(s, x(\eta(s))\big)\, ds\right| \\ &\leq |q(\theta(t_1)) - q(\theta(t_2))| + (1 - \|q\|)|\sigma(t_1) - \sigma(t_2)|. \end{aligned}$$

The continuity of θ, σ, q on the compact interval $[0,1]$ implies that $(B(x_n))_{n\in\mathbb{N}}$ is equi-continuous and then B is completely continuous by Arzela–Ascoli Lemma.

(b) *$F = AB$ satisfies the Furi–Pera condition.* For this purpose, consider a sequence $(x_j, \lambda_j)_{j\geq 1} \in \partial S \times [0,1]$ converging to some limit (x, λ) with $x = \lambda F(x)$ and $0 \leq \lambda < 1$. Then for j sufficiently large, we have

$$\|\lambda_j F(x_j)\| \leq \lambda \|F(x)\| + 1.$$

Since $x = \lambda F(x)$, we deduce the bounds:

$$\|x\| \leq \lambda \left(1 + \frac{\alpha R}{R+1}\|x\|\right),$$

and then

$$\|x\| \leq \frac{\lambda(R+1)}{R(1-\alpha\lambda)+1}\cdot$$

Hence

$$\|\lambda_j F(x_j)\| \leq \frac{R+1}{R(1-\alpha)+1}, \quad 0 \leq \lambda_j < 1.$$

This implies that, for R large enough, namely $R \geq \frac{1}{\sqrt{1-\alpha}}$, it holds that

$$\|\lambda_j F(x_j)\| \leq R$$

and so $\lambda_j F_j(x) \in S$. Finally

$$0 < k = \frac{R}{R+1} < \frac{1}{\|B(S)\|} = 1.$$

Then all assumptions of Theorem 3.1 are met with $C = 0$ and Equation (5) has a solution in X provided (6) holds true. Notice that for this first example, Corollary 3.2 may be applied as well; this will not be the case with the next two examples.

7.2 Example 2

(a) Consider the Banach space

$$X = C_0(\mathbb{R}, \mathbb{R}) = \{x \in C(\mathbb{R}, \mathbb{R}), \lim_{|t| \to +\infty} x(t) = 0\}$$

endowed with the sup-norm

$$\|x\|_X = \sup_{t \in \mathbb{R}} \{|x(t)|\}.$$

Let a continuous function $a\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that

$$a \text{ is } k\text{-Lipschitz with respect to the second argument} \tag{7}$$

and then define the mapping A by

$$Ax(t) = \int_{-|t|}^{|t|} e^{-\alpha |s|} a(s, x(s))\, ds, \quad t \in \mathbb{R},$$

for some positive parameter α. Then A is $\frac{2k}{\alpha}$-Lipschitz. Indeed

$$|Ax_1(t) - Ax_2(t)| \leq \left| \int_{-|t|}^{|t|} e^{-\alpha|s|} \Big(a(s, x_1(s)) - a(s, x_2(s)) \Big) ds \right|$$

$$\leq k \int_{-|t|}^{|t|} e^{-\alpha|s|} \, |x_1(s) - x_2(s)| \; ds \leq k_1 \|x_1 - x_2\| \int_{-|t|}^{|t|} e^{-\alpha|s|} \, ds.$$

Hence

$$\|Ax_1 - Ax_2\| \leq \frac{2k}{\alpha} \|x_1 - x_2\|.$$

Thus, we assume $\frac{2k}{\alpha} < 1$. In addition

$$\|Ax\| \leq \frac{2k}{\alpha} \|x\| + \frac{2}{\alpha} \|a(.,0)\|. \tag{8}$$

(b) Let the mapping B be defined by

$$Bx(t) = \int_{-\infty}^{+\infty} G(t,s) h\big(s, x(s)\big)\, ds,$$

where the nonlinear function $h\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and verifies the growth condition:

$$|h(t,x)| \leq q(t) \Psi(|x|), \quad \forall\, t, x \in \mathbb{R} \tag{9}$$

where $q \in C_0(\mathbb{R}, \mathbb{R}^+)$ and $\Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function. The kernel $G\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies

$$\exists \sigma, \varrho > 0, \; |G(t,s)| \leq \varrho e^{-\sigma|t-s|}, \quad \forall s, t \in \mathbb{R}. \tag{10}$$

We can show that B is completely continuous (see the proof of Theorem 2.1 in [8] for the details). Let the bounded closed and convex subset of X:

$$S = \{x \in X \colon \|x\| \leq R\},$$

where the positive constant R is to be selected later on.

(c) Assume that for any compact subset $K \subset \mathbb{R}$, there exists a positive constant $M_K > 0$ such that for any $x \in X$, $\lambda \in [0,1)$

$$x = \lambda AxBx \Longrightarrow (|x(t)| \leq M_K, \ \forall t \in K). \tag{11}$$

(d) To verify the Furi–Pera condition, let $(x_j, \lambda_j) \in \partial S \times [0,1]$ be such that, as $j \to +\infty$, $\lambda_j \to \lambda$ and $x_j \to x$ with $\lambda F(x)$ and $0 \leq \lambda < 1$. We show that $\lambda_j F(x_j) \in S$ where $F(x_j) = Ax_jBx_j$. Since Ψ is nondecreasing, we have

$$|Bx(t)| \leq \Psi(R) \int_{-\infty}^{+\infty} G(t,s)q(s)ds := \gamma(t).$$

Moreover, for each j, we have

$$\begin{aligned} \|\lambda_i F(x_j)\| &\leq \|Ax_j\| \cdot \|Bx_j\| \\ \leq \left(\frac{2k}{\alpha}\|x_1\| + \frac{2}{\alpha}\|a(\cdot,0)\|\right)\gamma(t) &\leq \left(\frac{2k}{\alpha}R + \frac{2}{\alpha}\|a(\cdot,0)\|\right)\gamma(t) \end{aligned}$$

Since $\lim_{t\to\pm\infty}\gamma(t) = 0$, there exist some t_1, t_2 and a sufficiently small positive constant M_1 such that

$$\|\lambda_i F(x_j)(t)\| \leq M_1, \quad \forall t \in (-\infty, t_1) \cup (t_2, +\infty). \tag{12}$$

In addition, for $t \in [t_1, t_2]$ and $x_j \in \partial S$, $\lim_{j\to+\infty} x_j = x = \lambda F(x)$ uniformly. Then for j large enough and $t \in [t_1, t_2]$, we have, from conditions (11) and (12), that

$$\|\lambda_i F(x_j)(t) \leq \lambda |F(x)(t)| + 1 \leq M_0 + 1. \tag{13}$$

Combining (12) and (13) and taking $R = \max(M_1, M_0 + 1)$, we get

$$\|\lambda_i F(x_j)(t)\| \leq R, \quad \forall t \in \mathbb{R}, \ \forall j \in \mathbb{N}.$$

Therefore the Furi–Pera condition is fulfilled.

As a consequence, we have proved that, under Assumptions (7), (9), (10) and (11), the nonlinear problem

$$\left(\int_{-|t|}^{|t|} e^{-\alpha|s|} a(s, x(s))\, ds\right)\left(\int_{-\infty}^{+\infty} G(t,s)h\big(s, x(s)\big)\, ds\right) = x(t), \quad t \in \mathbb{R}$$

admits, by Theorem 3.1, at least one solution $x \in \mathcal{B}_R(0) \subset C(\mathbb{R}, \mathbb{R})$.

Remark 7.1 Notice that in the particular case $a(.,0) \equiv 0$, the last condition in Theorem 3.1, namely $\|B(S)\|\frac{2k}{\alpha} < 1$, is equivalent to the Furi–Pera condition (see Remark 3.2). In such a case, we have only to find a function Ψ such that there exists some $R > 0$ such that

$$\frac{4k\varrho}{\alpha\sigma}\Psi(R)\|q\|_\infty < 1,$$

which is obviously satisfied whenever $\lim_{s\to+\infty} \Psi(s) = +\infty$.

7.3 Example 3

We will make use of the nonlinear version of Gronwall's Lemma (see [2])

Lemma 7.1 *Let $I = [a, b]$ and $u, g\colon I \to \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h\colon \mathbb{R} \to (0, +\infty)$ such that*

$$u(t) \le c + \int_a^t g(s)h(u(s))\,ds, \quad \forall t \in I.$$

Then, we have

$$u(t) \le \Psi^{-1}\left(\int_a^t g(s)\,ds\right), \quad \forall t \in I$$

with

$$\Psi(u) = \int_c^u \frac{dy}{h(y)}$$

for $u \ge c$ and Ψ^{-1} referring to the inverse of the function Ψ, provided for any $t \ge a$, $\int_a^t g(s)\,ds \in \operatorname{Dom}\Psi^{-1}$.

Let $a > 1$ and $X = C_0([a, +\infty), \mathbb{R})$ be the set of real continuous functions x defined on the interval $[a, +\infty)$ and such that $\lim_{t\to+\infty} x(t) = 0$. Equipped with the sup-norm $\|x\| = \sup_{t\ge a} |x(t)|$, it is a Banach space.

(a) On the space X, define a mapping A by

$$Ax(t) = h(t) + \int_a^t f(s, x(s))\,ds, \quad t \ge a$$

where $f\colon [a, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t, 0) = 0$, $t \ge a$ and there exists $p \in L^1([a, +\infty), \mathbb{R}^+)$ such that the nonlinear p-Lipschitz condition is satisfied:

$$|f(t, x(t)) - f(t, y(t))| \le p(t)|x(t) - y(t)|^\delta, \quad \forall (x, y) \in X^2, \tag{14}$$

with some $0 < \delta < 1$. The function $h\colon [a, +\infty) \to \mathbb{R}$ is continuous and nonidentically zero. If we let $|p|_1 = \int_a^{+\infty} p(t)dt$, then we can see that the operator A is $|p|_1$ $\mathcal{D}$-Lipschitzian and satisfies

$$\|Ax\| \le \|h\|_\infty + |p|_1\|x\|^\delta, \quad \forall x \in X. \tag{15}$$

(b) Define a second mapping B by

$$Bx(t) = \sigma(t)\phi(x(t)) + \int_a^t G(t,s)g\big(s,x(s)\big)\,ds, \quad t \geq a$$

with continuous functions $\sigma\colon [a,+\infty) \to \mathbb{R}$ and $g\colon [a,+\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying respectively $\lim_{t\to+\infty} \sigma(t) = 0$ and the growth assumption

$$|g(s,\xi)| \leq q(s)\psi(|\xi|), \quad \forall (s,x) \in [a,+\infty) \times \mathbb{R},$$

where $q \in L^1([a,+\infty),\mathbb{R}^+)$ and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing map. The kernel function $G\colon [a,+\infty)^2 \to \mathbb{R}$ satisfies

$$\limsup_{t\to+\infty} \int_a^{+\infty} |G(t,s)|q(s)\,ds = 0. \tag{16}$$

Finally, let

$$\overline{\sigma} = \sup_{t\geq a} |\sigma(t)| \quad \text{and} \quad \overline{\alpha} = \sup_{t\geq a} \alpha(t)$$

with

$$\alpha(t) = \int_a^{+\infty} |G(t,s)|q(s)\,ds.$$

With conditions (16), we can show, as in the proof of Theorem 2.1 in [8], that B is completely continuous. Moreover it satisfies

$$\|Bx\| \leq \overline{\sigma}\phi(\|x\|) + \overline{\alpha}\psi(\|x\|), \quad \forall\, x \in X. \tag{17}$$

(c) Assume that for any compact subset $K \subset \mathbb{R}$, there exists a positive constant $M_K > 0$ such that for any $x, y \in X$, and $\lambda \in [0,1)$,

$$y = \lambda AyBx \Longrightarrow (|y(t)| \leq M_K,\, \forall\, t \in K). \tag{18}$$

Then, it remains to check the Furi–Pera condition (c) in Theorem 3.3 for the mapping N defined by $Nx = y = AyBx$ with respect to a closed ball $S = \overline{\mathcal{B}}_R(0)$ for some positive constant R. Let $(x_j,\lambda_j) \in \partial S \times [0,1]$ be a sequence such that, as $j \to +\infty$, $\lambda_j \to \lambda$ and $x_j \to x$ with $x = \lambda N(x)$ and $0 \leq \lambda < 1$. We will show that $\lambda_j N x_j \in S$. Let $y_j = Nx_j = Ay_jBx_j$. Then $\lambda_j N x_j = \lambda_j A y_j B x_j = \lambda_j y_j$. To perform an estimate of $|y_j|$, write

$$y_j(t) = h(t)Bx_j(t) + Bx_j(t)\int_a^t f(s,y_j(s))\,ds$$

and notice that, as in (17)

$$|Bx_j(t)| \leq |\sigma(t)|\phi(R) + \alpha(t)\psi(R) := \gamma_R(t) \leq \overline{\gamma}_R, \quad t \geq a,$$

where $\overline{\gamma}_R = |\overline{\sigma}|\phi(R) + \overline{\alpha}\psi(R)$ and

$$\lim_{t\to+\infty} \gamma_R(t) = 0. \tag{19}$$

Then

$$|y_j(t)| \le \gamma_R(t)\left(\|h\|_\infty + \int_a^t p(s)|y_j(s)|^\delta\, ds\right)$$

which yields

$$\frac{|y_j(t)|}{\gamma_R(t)} \le \|h\|_\infty + (\overline{\gamma}_R)^\delta \int_a^t p(s)\left|\frac{y_j(s)}{\gamma_R(s)}\right|^\delta ds.$$

By Lemma 7.1, we deduce the upper bound

$$|y_j(t)| \le \gamma_R(t)\Psi^{-1}\left((\overline{\gamma}_R)^\delta \int_a^t p(s)ds\right),$$

where

$$\Psi(u) = \int_{\|h\|_\infty}^u s^{-\delta}ds.$$

With (19), it follows that there exist $\overline{R} > 0$ and $t_1 > a$ such that

$$|y_j(t)| \le \overline{R}. \tag{20}$$

This both with (18) enable us to distinguish between the cases t in a compact subset of $[a, +\infty)$ and t large enough and prove, as in example 2, that there exists some $R > 0$ such that $\|y_j\| \le R$. Therefore $\lambda_j N x_j$ belong to S proving our claim follows.

Finally, Assumption $(\mathcal{H})$ in Theorem 3.3 is verified for

$$\begin{aligned}\Phi(r) &= r - \|B(S)\|\varphi_A(r) \ge r - [\overline{\sigma}\phi(R) + \overline{\alpha}\psi(R)]\,|p|_1 r^\delta \\ &= r\left(1 - [\overline{\sigma}\phi(R) + \overline{\alpha}\psi(R)]\,|p|_1 r^{\delta-1}\right)\end{aligned} \tag{21}$$

which increases to positive infinity as $r \to +\infty$ for $0 < \delta < 1$. To sum up, we have proved that, under the hypotheses on f, g, and h the nonlinear integral equation

$$x(t) = \left(h(t) + \int_a^t f(s, x(s))\, ds\right)\left(\int_a^t g\big(s, x(s)\big)\, ds\right), \quad t \in [a, +\infty)$$

has a solution $x \in C_0([a, +\infty), \mathbb{R})$ by Theorem 3.3.

8 Concluding remarks

(a) The following functional integral equation

$$x(t) = \left(\frac{1}{1 + |x(\theta(t))|}\right)\left(q(t) + \int_0^{\sigma(t)} g\big(s, x(\eta(s))\big)\, ds\right) \tag{22}$$

is discussed in [10] and solutions are proven to exist in the unit ball $S = \mathcal{B}_1(0)$ under the assumptions of Theorem 1.3. Indeed, notice that all solutions of

Equation (22) are in S. By contrast, solutions x of Equation (5) satisfy $\|x\| \leq R+1$ and thus do not lie in $S = \mathcal{B}_R(0)$. As a consequence, Theorem 1.3 could not be used to solve Equation (5) though the Furi–Pera condition was satisfied and Theorem 3.1 was successfully applied in Example 1.

(b) If, in Example 3, we have rather applied Theorem 3.1 instead, we should be led to the following estimates regarding the verification of the Furi–Pera condition for the mapping $F = AB$:

$$\begin{aligned}\|Fx_j\| \leq \|Ax_j\|\|Bx_j\|| \leq \left(|h\|_\infty + |p|_1\|x_j\|^\delta\right)(\overline{\sigma}\phi(\|x_j\|) + \overline{\alpha}\psi(\|x_j\|)) \\ \leq (|h\|_\infty + |p|_1 R)(\overline{\sigma}\phi(R) + \overline{\alpha}\psi(R)).\end{aligned}$$

Therefore the Furi–Pera condition is satisfied whenever

$$\left(|h\|_\infty + |p|_1 R^\delta\right)(\overline{\sigma}\phi(R) + \overline{\alpha}\psi(R)) \leq R. \tag{23}$$

This inequality is somewhat restrictive and shows that the Schauder's fixed point theorem could be applied as well.

(c) As illustrated by examples 1-3, Theorems 3.1 and 3.3 show that, in practice, the Furi–Pera condition is easier to be used than the condition (c) in Theorem 1.3. Indeed, we can notice that assumption (c) in Examples 2 and 3 is a weak condition in the sense that we were able to make use of the Furi–Pera condition while neither Schauder's fixed point theorem nor Dhage's fixed point theorem could be applied. Moreover, fixed point theorems in Banach algebras are useful in applications when problems exhibit nonlinearities as the product of two integral functions. Many boundary value problem for second-order and higher-order nonlinear differential equations may be reduced to integral equations. Further to Examples 1-3, we refer for instance to the functional integral equations treated in [12, 13].

Acknowledgment The first author is grateful to Laboratory EDP&HM of École Normale Supérieure, Kouba in Algiers, for its constant support.

References

[1] Agarwal, R. A., Meehan, M., O'Regan, D.: Fixed Point Theory and Applications. Cambridge Tracts in Mathematics, Vol. 141, *Cambridge University Press*, 2001.

[2] Bainov, D., Simeonov, P.: Integral Inequalities and Applications. *Kluwer Academic Publishers*, 1992.

[3] Banas, J., Goebel, K.: Measure of Noncompactness in Banach Spaces. Math. and Appli., Vol. 57, *Marcel Dekker, New York*, 1980.

[4] Banas, J., Knap, Z.: *Measure of noncompactness and nonlinear integral equations of convolution type.* J. Math. Anal. Appl. **146** (1990), 353–362.

[5] Boyd, D. W., Wong, J. S. W.: *On nonlinear contractions.* Proc. Amer. Math. Soc. **20** (1969), 458–464.

[6] Burton, T. A.: *A fixed point theorem of Krasnozels'kĭi.* Appl. Math. Lett. **11** (1998), 85–88.

[7] Deimling, K.: Nonlinear Functional Analysis. *Springer Verlag, Berlin–Tokyo*, 1985.

[8] Djebali, S., Moussaoui, T.: *A class of second order BVPs on infinite intervals.* Elec. Jour. Qual. Theo. Diff. Eq. **4** (2006), 1–19.

[9] Dhage, B. C.: *On some variants of Schauder's fixed point principle and applications to nonlinear integral equations.* Jour. Math. Phys. Sci. **2**, 5 (1988), 603–611.

[10] Dhage, B. C.: *On a fixed point theorem in Banach algebras with applications.* Appl. Math. Lett. **18** (2005), 273–280.

[11] Dhage, B. C.: *Some nonlinear alternatives in Banach algebras with applications I.* Nonlinear Studies **12**, 3 (2005), 271–281.

[12] Dhage, B. C., Ntouyas, S. K.: *Existence results for nonlinear functional integral equations via a fixed point theorem of Krasnozels'kĭi–Schaefer type.* Nonlinear Studies **9**, 3 (2002), 307–317.

[13] Dhage, B. C., O'Regan, D.: *A fixed point theorem in Banach algebras with applications to functional integral equations.* Funct. Diff. Equ. **7**, 3-4 (2000), 259–267.

[14] Furi, M., Pera, P.: *A continuation method on locally convex spaces and applications to ODE on noncompact intervals.* Annales Polonici Mathematici **47** (1987), 331–346.

[15] Krasnozels'kĭi, M. A.: Positive Solutions of Operator Equations. *Noordhoff, Groningen*, 1964.

[16] Krasnozels'kĭi, M. A.: Integral Operators in Space of Summable Functions. *Noordhoff, Leyden*, 1976.

[17] O'Regan, D.: *Fixed-point theory for the sum of two operators.* Appl. Math. Lett. **9** (1996), 1–8.

[18] Smart, D. R.: Fixed point Theorems. *Cambridge University Press*, 1974.

[19] Zeidler, E.: Nonlinear Functional Analysis and its Applications. Vol. I: Fixed Point Theorems. *Springer Verlag, New York*, 1986.