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Further Results on Global Stability of Solutions of Certain Third-order Nonlinear Differential Equations

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Abstract

Sufficient conditions are established for the global stability of solutions of certain third-order nonlinear differential equations. Our result improves on Tunc's [10].

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1 Introduction

We consider the third-order nonlinear ordinary differential equation

$$\dddot{x} + \psi(x, \dot{x}, \ddot{x})\ddot{x} + f(x, \dot{x}) = 0 \tag{1.1}$$

or its equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -\psi(x, y, z)z - f(x, y), \tag{1.2}$$

where

$$\psi, \psi_x, \psi_z \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \quad \text{and } f, f_x, f_y \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}). \tag{1.3}$$

It is assumed that solutions of (1.1) exist and are unique.

Stability is a very important problem in the theory and application of differential equations, and an effective method for studying the stability of nonlinear differential equations is the second method of Lyapunov (see [1–14]).

In a recently paper, Tunc [10] obtained the global stability of (1.1) and the following result was proved.

Theorem A (Tunc [10]). *Further to the basic assumptions on the functions* ψ *and* f *suppose the following:*

(i) $xf(x,0) > 0$ *for* $x \neq 0$*;*

(ii) $\int_0^y f(0,v)\,dv \geq 0$*;*

(iii) $\lim_{|x|\to\infty} \sup \int_0^x f(u,0)\,du = \infty$*;*

(iv) there is a positive constant B *such that* $\psi(x,y,z) \geq B$ *for all* x, y, z*;*

(v)

$$B\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v\,dv\right] y \geq y \int_0^y f_x(x,v)\,dv$$

for all x, y*;*

(vi)

$$B\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v\,dv\right] y + \psi(x,y,z) \geq y \int_0^y f_x(x,v)\,dv + B$$

for all $x, y \neq 0, z$*;*

(vii)

$$4B \int_0^x f(u,0)\,du \left\{\int_0^y [f(x,v) - f(x,0)]\,dv + B \int_0^y [\psi(x,v,0) - B]\,v\,dv\right\} \geq y^2 f^2(x,0)$$

for all $x, y \neq 0$*;*

(viii) $y\psi_z(x,y,z) \geq 0$ *for all* x, y, z

Then the trivial solution of equation (1.1) is globally asymptotically stable.

Interestingly, (1.1) is a rather general third-order nonlinear differential equation. In particular, many third-order differential equations which have been discussed in [12] are special cases of (1.1), and some known results can be obtained using this theorem. However, it is not easy to apply Theorem A to these special cases to obtain new or better results since Theorem A has some hypotheses which are not necessary for the stability of many nonlinear equations.

Our aim in this paper is to further study the global stability of (1.1). In the next section, we establish a criterion for the stability of (1.1), which extends and improve Theorem A. Finally, in Section 3, we apply our result to some examples.

In the following discussion, we always assume (1.3) holds without further mention.

2 Main result

Our main result in this section is the following theorem.

Theorem *Let δ_0, a, b, c be positive constants such that $ab > c$. Assume that*

(1) $\frac{f(x,0)}{x} \geq \delta_0, \quad x \neq 0, \quad f(0,0) = 0,$

(2) $f'(x,0) \leq c,$

(3) $f_y(x, \theta y) \geq b$ *for* $0 \leq \theta \leq 1,$

(4) $\psi(x,y,z) > a,$

(5) $y\psi_z(x,y,\theta z) \geq 0,$ *for* $0 \leq \theta \leq 1,$

(6) $a\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v\,dv\right] y \geq y \int_0^y f_x(x,v)\,dv.$

Then, the trivial solution of (1.1) is globally asymptotically stable.

Remark 1 The theorem just stated above improves the theorem established in [1] and includes the result established in [9]. The results of Ezeilo [2], Ogurtsov [5] and Goldwyn and Narendra [3] are also direct consequences of our result.

Proof Clearly, (1.1) is equivalent to the system (1.2) and $(0,0,0)$ is a solution. Now, consider the Lyapunov function

$$V(x,y,z) = \int_0^x f(u,0)\,du + \int_0^y \psi(x,v,0)v\,dv + a^{-1}\int_0^y f(x,v)\,dv + \frac{1}{2}a^{-1}z^2 + yz \tag{2.1}$$

This is rewritten as

$$V(x,y,z) = \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}(f(x,0)+by)^2 + \int_0^y [\psi(x,v,0)-a]v\,dv + \frac{1}{a}\int_0^y [f_v(x,\theta v) - b]v\,dv + \int_0^x [1 - \frac{1}{ab}f'(u,0)]f(u,0)\,du$$

where $f_v(x,\theta v) = v^{-1}\{f(x,v) - f(x,0)\}$, $v \neq 0$.

On using hypotheses (1)–(4) of the theorem,

$$V(x,y,z) \geq \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}(f(x,0)+by)^2 + \frac{1}{2}\delta_1 x^2,$$

where $\delta_1 = \frac{1}{ab}(ab-c)\delta_0 > 0$. It follows that there exists a constant $K > 0$ small enough that

$$V(x,y,z) \geq K(x^2+y^2+z^2).$$

Hence $V(x,y,z)$ is a positive definite function.

Next, we show that the derivative of $V(x,y,z)$ with respect to t along the solution path of (1.2) is negative semi definite.

$$\dot{V}_{(1.2)} = V_x\dot{x} + V_y\dot{y} + V_z\dot{z}, \tag{2.2}$$

where V_x, V_y, V_z are partial derivatives of V with respect to x, y and z respectively, and $\dot{x}, \dot{y}$ and $\dot{z}$ are as in (1.2).
Thus,

$$\begin{aligned} V_x &= f(x,0) + \int_0^y \psi_x(x,v,0)v\,dv + \frac{1}{a}\int_0^y f_x(x,v)\,dv, \\ V_y &= \psi(x,y,0)y + \frac{1}{a}f(x,y) + z, \qquad V_z = \frac{1}{a}z + y. \end{aligned}$$

Then, substituting V_x, V_y, V_z in (2.2) and using (1.2) yield

$$\begin{aligned} \dot{V}_{(1.2)}(x,y,z) &= -\left\{ f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v\,dv \right\} y \\ &+ \frac{1}{a}y\int_0^y f_x(x,v)\,dv - \psi_z(x,y,\theta z)yz^2 - \left[\frac{1}{a}\psi(x,y,z) - 1\right] z^2, \end{aligned}$$

where $\psi_z(x,y,\theta z) = z^{-1}\{\psi(x,y,z) - \psi(x,y,0)\}$, $z \neq 0$. From hypotheses (4), (5) and (6) of theorem, we see that

$$\dot{V}_{(1.2)}(x,y,z) \leq 0, \tag{2.3}$$

and the rest of the proof may now follow as in [2, 9].

Let Ω denote a trajectory $x(t), y(t), z(t)$ of (1.2) satisfying the initial conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, where (x_0, y_0, z_0) is an arbitrary point of the (x,y,z)-space. Then, by (2.3),

$$V(t) \equiv V(x(t), y(t), z(t)) \leq V(x_0, y_0, z_0) \qquad (t \geq 0). \tag{2.4}$$

Further, $V(t)$, being non-increasing and non-negative, tends to a non-negative limit, $V(\infty)$ say, as $t \to \infty$. To prove the theorem, it is sufficient to show that

$$V(\infty) \not> 0; \tag{2.5}$$

for, in that event, we should have $V(\infty) = 0$, and this would imply $x(\infty) = 0$, $y(\infty) = 0$, $z(\infty) = 0$, which is the required result.

Suppose on the contrary that (2.5) is not true: that is, assume that $V(\infty) > 0$. Since the set points (x, y, z) for which

$$V(x, y, z) \leq V(x_0, y_0, z_0)$$

is bounded, it is clear from (2.4) that the trajectory Ω has limit points; and the set of all its limit points consists of whole trajectories of (1.2) lying on the surface $V(x, y, z) = V(\infty)$. Thus, in particular, if Q is a limit point of Ω, there is a half-trajectory, Ω_Q say, of (1.2) issuing from Q and lying on the surface $V(x, y, z) = V(\infty)$. Evidently, we must have

$$\dot{V}_{(1.2)} \equiv 0 \tag{2.6}$$

on Ω_Q; for otherwise there would exist points (x, y, z) of Ω_Q at which

$$V(x, y, z) < V(\infty).$$

From (2.4) and (2.6) it follows readily that $z = 0$ and hence also that $y \equiv \gamma$, $x = \gamma t + \xi$ (γ, ξ constants), $\dot{z} = 0$ for any (x, y, z) on Ω_Q.

Also, since from (1.2),

$$\dot{z} = -\psi(x, y, z)z - f(x, y),$$

it follows that $f(x, y) = 0$, that is

$$f(\gamma t + \xi, \gamma) = 0. \tag{2.7}$$

Since $f(0, 0) = 0$, (2.7) clearly holds if and only if $\xi = \gamma = 0$ (see, for example, [12, p. 370]). Hence $x = 0$, $y = 0$. We have therefore that $x = y = z = 0$; this implies that the origin is a point of the surface $V(x, y, z) = V(\infty)$, which contradicts our assumption that $V(\infty) > 0$. This proves (2.5) and hence the theorem. □

Remark 2 Clearly our theorem is an improvement and extension of Theorem A. In particular, from our theorem we see that (ii), (vi) and (viii) assumed in Theorem A are not necessary, and (i) can be replaced by (1) for the global stability of the trivial solution of (1.1).

3 Examples

In this section, we consider certain examples which are particular cases of (1.1).

Example 1 Consider the equation

$$\dddot{x} + [(\sin x)\dot{x} + (\dot{x})^2 + e^{\dot{x}\ddot{x}} + 2]\ddot{x} + (\dot{x})^3 + \dot{x} + \frac{x}{1 + x^2} = 0. \tag{3.1}$$

(3.1) is in the form of (1.1) with

$$\psi(x, y, z) = (\sin x)y + y^2 + e^{yz} + 2, \qquad f(x, y) = y^3 + y + \frac{x}{1 + x^2}.$$

With $a = 2$, $b = 1$, $c = 1$, we observe that

$$\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v dv\right] y = \left[y^3 + y - \frac{1}{3}(\cos x)y^3\right] y$$
$$> y^2 \frac{1-x^2}{(1+x^2)^2} = y^2 f'(x,0), \quad \text{for } y \neq 0.$$

Then it is easy to check all the hypotheses in Theorem are satisfied and so the trivial solution of (3.1) is globally asymptotically stable.

Example 2 Consider the equation

$$\dddot{x} + [\ln(1+x^2) + e^{\dot{x}\ddot{x}} + 2] + \frac{x}{1+x^2}(1+(\dot{x})^2) + \dot{x} + \frac{1}{3}(\dot{x})^3 = 0. \tag{3.2}$$

(3.2) is in the form (1.1) with

$$\psi(x,y,z) = \ln(1+x^2) + e^{yz} + 2, \qquad f(x,y) = \frac{x}{1+x^2}(1+y^2) + y + \frac{1}{3}y^3.$$

With $a = 2$, $b = 1$, $c = 1$, we observe that

$$\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v,0)v\,dv\right] y = \left[y + \frac{1}{3}y^3\right] y$$
$$> y^2 \frac{(1-x^2)}{(1+x^2)^2} = y^2 f'(x,0), \quad \text{for } y \neq 0.$$

Then it is easy to check all the hypotheses in Theorem are satisfied and so the trivial solution of (3.2) is globally asymptotically stable.

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