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On Weakly and Pseudo Concircular Symmetric Structures on a Riemannian Manifold

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Abstract

In this paper, we examine the properties of hypersurfaces of weakly and pseudo concircular symmetric manifolds and we give an example for these manifolds.

Key words: Weakly symmetric manifold, pseudo symmetric manifold, weakly and pseudo symmetric concircular manifold, totally umbilical, totally geodesic, mean curvature, scalar curvature.

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1 Introduction

Firstly, Tamassy and Binh introduced weakly symmetric manifolds, [1].

A non-flat Riemannian manifold (M_n, g), $(n > 2)$ whose the curvature tensor satisfies the following relation is called weakly symmetric

$$\nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lijk} + D_i R_{hljk} + E_j R_{hilk} + F_k R_{hijl} \tag{1.1}$$

where A, B, D, E, F are non-zero 1-forms and ∇ denotes the covariant differentiation with respect to the metric tensor of the manifold. These 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. It may be mentioned in this connection that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric space studied by Chaki and Mondal, [2], the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo-symmetric manifold. De and Bandyopadhyay, [3], proved that 1-forms of $(WS)_n$ can not be all different. Then the equation (1.1) reduces to the form

$$\nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl} \tag{1.2}$$

Let us consider a subspace V_m immersed in a Riemannian manifold V_n whose parametric representation is $u^\lambda = u^\lambda(u^1, u^2, \ldots, u^m)$ where (u^λ) and (u^i) $(i, j, k, \ldots = 1, 2, \ldots, m)$ denote the coordinate systems of V_n and V_m, respectively. A conformal transformation $\bar{g}_{ij} = \rho^2 g_{ij}$ of the fundamental tensor of V_n, being a concircular one with the function ρ satisfying the equations

$$\rho_{ij} = \nabla_j \rho_i - \rho_i \rho_j + \frac{1}{2} g^{\alpha\beta} \rho_\alpha \rho_\beta g_{ij} = \phi g_{ij}, \qquad \rho_j = \frac{\partial}{\partial u^j} \ln \rho \tag{1.3}$$

this transformation is called concircular transformation where ϕ is a function of u^i.

The present paper deals with non-concircular flat Riemannian manifold (M_n, g) whose concircular curvature tensor Z_{hijk} satisfies the condition $(n > 2)$

$$\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lijk} + D_i Z_{hljk} + E_j Z_{hilk} + F_k Z_{hijl}$$

where

$$Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{hk} g_{ij} - g_{hj} g_{ik})$$

R_{hijk} is the curvature tensor and R is the scalar curvature. Such a manifold will be called a weakly concircular symmetric manifold and denoted by $(WZS)_n$, [4]. It was shown that, in [5], Z^h_{ijk} is invariant under a concircular transformation.

Desa and Amur studied the concircular recurrent Riemannian manifold, [6]. The authors proved that the defining condition of a $(WZS)_n$ can always be expressed in the following form, [4]

$$\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lijk} + B_i Z_{hljk} + D_j Z_{hilk} + D_k Z_{hijl} \tag{1.4}$$

where A, B, D 1-forms (non-zero simultaneously).

From the first Bianchi identity, we get

$$R_{hijk} + R_{hjki} + R_{hkij} = 0 \tag{1.5}$$

The second Bianchi identity for a Riemannian manifold is

$$\nabla_s R_{hijk} + \nabla_j R_{hiks} + \nabla_k R_{hisj} = 0 \tag{1.6}$$

Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U, y^\alpha\}$. Let (M, g) be a hypersurface of $(\bar{M}, \bar{g})$ defined via a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take the values $1, 2, \dots, n+1$ and Latin indices take the values $1, 2, \dots, n$ a locally coordinate system. Then, we have

$$g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta \tag{1.7}$$

Let n^α be a local unit normal to (M, g). Thus, we obtain $\bar{g}_{\alpha\beta} n^\alpha y_i^\beta = 0$, $g_{\alpha\beta} n^\alpha n^\beta = 1$ and it is easily seen that there are the following conditions between the contrary metric tensors of the hypersurface (M, g) and $(\bar{M}, \bar{g})$

$$g^{\alpha\beta} = g^{ij} y_i^\alpha y_j^\beta + n^\alpha n^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}, \quad (i, j = 1, 2, \dots, n;\ \alpha = \beta = 1, 2, \dots, n+1) \tag{1.8}$$

A point of a hypersurface, at which the principal directions of the curvature are indeterminate, is called an umbilical point. In order that the lines of curvature may be indeterminate at every point of the hypersurface, it is necessary and sufficient that $\Omega_{ij} = \omega g_{ij}$, where ω is an invariant. According to [7],

$$M = \Omega_{ij} g^{ij} = n\omega \tag{1.9}$$

where the scalar M is called the mean curvature of such a hypersurface, so that the conditions for indeterminate lines of curvature are expressible as

$$\Omega_{ij} = \frac{M}{n} g_{ij} \tag{1.10}$$

If all the geodesics of a hypersurface (M, g) are also geodesics of $(\bar{M}, \bar{g})$, the former is called a totally geodesic hypersurface of the latter. Such hypersurfaces are generalizations of planes in ordinary space. A necessary and sufficient condition that (M, g) be a totally geodesic hypersurface is that the normal curvature should vanish for all directions in (M, g), and at every point. This requires

$$\Omega_{ij} = 0 \tag{1.11}$$

Consequently,

$$M = 0 \tag{1.12}$$

and (1.10) is satisfied.

The structure equations of Gauss and Mainardi–Codazzi, [8]

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\theta} B_{ijkl}^{\alpha\beta\gamma\theta} + \Omega_{ijkl}$$

and

$$\nabla_k \Omega_{ij} - \nabla_j \Omega_{ik} + \bar{R}_{\beta\gamma\delta\theta} n^\beta B_{ijk}^{\gamma\delta\theta} = 0$$

where $\Omega_{ijkl} = \Omega_{lj}\Omega_{ik} - \Omega_{il}\Omega_{jk}$.

From (1.9), the above equations reduce to the following forms

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\theta} B_{ijkl}^{\alpha\beta\gamma\theta} + \frac{M^2}{n^2}(g_{lj} g_{ik} - g_{li} g_{jk}) \tag{1.13}$$

and

$$\bar{R}_{\alpha\gamma\delta\theta}n^{\alpha}B^{\gamma\delta\theta}_{ijk} = \frac{1}{n}(g_{ik}\nabla_j M - g_{ij}\nabla_k M) \tag{1.14}$$

respectively, where R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\theta}$ are the curvature tensors (M, g) and $(\bar{M}, \bar{g})$, and $B^{\alpha\beta\gamma\theta}_{ijkl} = B^{\alpha}_i B^{\beta}_j B^{\gamma}_k B^{\theta}_l$, $B^{\alpha}_i = y^{\alpha}_i$.

From the Gauss equation, we get

$$\bar{R} = R + 2\bar{R}_{\alpha\beta}n^{\alpha}n^{\beta} - \Omega_{ijkl}g^{il}g^{jk} \tag{1.15}$$

The concircular curvature tensors of (M, g) and $(\bar{M}, \bar{g})$ can be written in the form

$$Z_{hijk} = R_{hijk} + \frac{R}{n(n-1)}G_{hijk} \tag{1.16}$$

and

$$\bar{Z}_{\alpha\beta\gamma\theta} = \bar{R}_{\alpha\beta\gamma\theta} + \frac{\bar{R}}{n(n+1)}G_{\alpha\beta\gamma\theta} \tag{1.17}$$

where $G_{hijk} = g_{hj}g_{ik} - g_{hk}g_{ij}$ and $G_{\alpha\beta\gamma\theta} = \bar{g}_{\alpha\gamma}\bar{g}_{\beta\theta} - \bar{g}_{\alpha\theta}\bar{g}_{\beta\gamma}$. On account of (1.7), (1.13), (1.16) and (1.17), we get

$$Z_{hijk} = \bar{Z}_{\alpha\beta\gamma\theta}B^{\alpha\beta\gamma\theta}_{hijk} + \frac{M^2}{n^2}G_{hijk} + \frac{1}{n}(\frac{R}{n-1} - \frac{\bar{R}}{n+1})G_{hijk} \tag{1.18}$$

2 Totally umbilical hypersurface of a weakly concircular symmetric manifold

Now, we consider an $(n+1)$-dimensional weakly concircular symmetric Riemannian manifold and we denote this manifold by $(WZS)_{n+1}$. For a $(WZS)_{n+1}$, we have

$$\nabla_e\bar{Z}_{abcd} = A_e\bar{Z}_{abcd} + B_a\bar{Z}_{ebcd} + B_b\bar{Z}_{aecd} + D_c\bar{Z}_{abed} + D_d\bar{Z}_{abce} \tag{2.1}$$

Using (1.17), we obtain

$$\bar{Z}_{abcd}n^a B^{bcd}_{ijk} = \bar{R}_{abcd}n^a B^{bcd}_{ijk} \tag{2.2}$$

We assume that the scalar curvature of $(WZS)_n$ is not constant and $(WZS)_n$ is a totally umbilical hypersurface. In this case, we find that

$$\begin{aligned}\nabla_s Z_{hijk} &= A_s\bar{Z}_{abcd}B^{abcd}_{hijk} + B_h\bar{Z}_{ebcd}B^{ebcd}_{sijk} + B_i\bar{Z}_{aecd}B^{aecd}_{hsjk}\\ &+ D_j\bar{Z}_{abed}B^{abed}_{hisk} + D_k\bar{Z}_{abce}B^{abce}_{hijs} + \frac{1}{n^2}G_{hijk}\nabla_s M^2\\ &+ \frac{1}{n}G_{hijk}\nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big) + \frac{M}{n}\Big(g_{hs}\bar{R}_{abcd}B^{bcd}_{ijk}n^a + g_{is}\bar{R}_{badc}B^{adc}_{hkj}n^b\\ &+ g_{js}\bar{R}_{cdab}B^{dab}_{khi}n^c + g_{ks}\bar{R}_{dcba}B^{cba}_{jih}n^d\Big)\end{aligned} \tag{2.3}$$

By the aid of the Gauss equation, (2.3) can be written as

$$\begin{aligned}
\nabla_s Z_{hijk} = A_s\Big(&Z_{hijk} - \frac{M^2}{n^2}G_{hijk} - \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)G_{hijk}\Big)\\
&+ B_h\Big(Z_{sijk} - \frac{M^2}{n^2}G_{sijk} - \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)G_{sijk}\Big)\\
&+ B_i\Big(Z_{hsjk} - \frac{M^2}{n^2}G_{hsjk} - \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)G_{hsjk}\Big)\\
&+ D_j\Big(Z_{hisk} - \frac{M^2}{n^2}G_{hisk} - \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)G_{hisk}\Big)\\
&+ D_k\Big(Z_{hijs} - \frac{M^2}{n^2}G_{hijs} - \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)G_{hijs}\Big)\\
&+ \frac{1}{n^2}G_{hijk}\nabla_s M^2 + \frac{1}{n}G_{hijk}\nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\\
&+ \frac{M}{n^2}[(g_{hs}g_{ik} - g_{is}g_{hk})\nabla_j M + (g_{is}g_{hj} - g_{ij}g_{hs})\nabla_k M\\
&+ (g_{js}g_{ik} - g_{ij}g_{sk})\nabla_h M + (g_{ks}g_{hj} - g_{js}g_{hk})\nabla_i M] \qquad (2.4)
\end{aligned}$$

Now, we suppose that (M, g) is $(WZS)_n$.
By the aid of (1.4) and (2.4), we have

$$\begin{aligned}
\Big[\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\Big]&(A_s G_{hijk} + B_h G_{sijk} + B_i G_{hsjk} + D_j G_{hisk} + D_k G_{hijs})\\
&- G_{hijk}\nabla_s\left(\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\right)\\
- \frac{M}{n^2}(G_{hisk}\nabla_j M + G_{ihsj}\nabla_k M &+ G_{sijk}\nabla_h M + G_{kjsh}\nabla_i M) = 0 \qquad (2.5)
\end{aligned}$$

Multiplying (2.5) by $g^{hk}g^{ij}$, we can obtain

$$\begin{aligned}
&\left(\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\right)(2B_s + 2D_s + nA_s)\\
&- \frac{(n+2)}{n^2}\nabla_s M^2 - \nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big) = 0 \qquad (2.6)
\end{aligned}$$

Similarly, multiplying (2.5) by $g^{ik}g^{hs}$, it is easily obtained that

$$\begin{aligned}
&\left(\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\right)(B_s + A_s + (n-1)D_s)\\
&- \frac{(n+2)}{2n^2}\nabla_s M^2 - \frac{1}{n}\nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big) = 0 \qquad (2.7)
\end{aligned}$$

Let us suppose that

$$R = (1 - \frac{2}{n+1})\bar{R} \qquad (2.8)$$

where the scalar curvature R is not constant.

From (2.6) and (2.7), we get

$$A_s = 2D_s \quad \text{or} \quad M = 0 \tag{2.9}$$

We assume that $A_s = 2D_s$. Transvecting (1.4) with g^{lk} and g^{ij}, we get

$$g^{ks}\nabla_s G_{hk} = (A_k - B_k + D_k)G_{hs}g^{sk} \tag{2.10}$$

where $G_{hk} = R_{hk} - \frac{R}{n}g_{hk}$ $(n > 2)$ is the Einstein tensor.

Similarly, transvecting (1.4) with g^{hk} and g^{ij}, we have

$$(B_k + D_k)G_{hs}g^{sk} = 0 \tag{2.11}$$

Hence, using the equations $(2.9)_1$ and (2.10), it can be obtained that

$$(A_k + 2B_k)G_{hs}g^{sk} = 0 \tag{2.12}$$

Now, multiplying the equation (1.4) by g^{hl} and g^{ij} and using the result $\nabla_s R^s_h = \frac{1}{2}\nabla_h R$, we obtain $R \equiv$ const. In the beginning, we suppose that $R \neq const$. Thus, $A_s \neq 2D_s$. From (2.9), we have $M = 0$, i.e., the hypersurface is totally geodesic. Thus, we can state the following theorem:

Theorem 2.1 *In the totally umbilical hypersurface $(WZS)_n$ of $(WZS)_{n+1}$, if the expression $R = (1 - \frac{2}{n+1})\bar{R}$, $(R \neq const.)$ is satisfied then the hypersurface is totally geodesic.*

Theorem 2.2 *If the totally umbilical hypersurface $(WZS)_n$ of a $(WZS)_{n+1}$ satisfies the condition $\frac{R}{n-1} - \frac{\bar{R}}{n+1} = c$ $(c < 0,\ const.)$ then either the mean curvature or the scalar curvature of this hypersurface is constant.*

Proof We assume that the totally umbilical hypersurface $(WZS)_n$ of $(WZS)_{n+1}$ satisfies the condition

$$-\frac{\bar{R}}{n+1} + \frac{R}{n-1} = c \tag{2.13}$$

From (2.5) and (2.13), we obtain

$$\begin{aligned}\Big(\frac{M^2}{n^2} + \frac{c}{n}\Big)&(A_s G_{hijk} + B_h G_{sijk} + B_i G_{hsjk} + D_j G_{hisk} + D_k G_{hijs})\\ &- \frac{1}{n^2}G_{hijk}\nabla_s M^2 - \frac{M}{n^2}(G_{hisk}\nabla_j M\\ &+ G_{ihsj}\nabla_k M + G_{sijk}\nabla_h M + G_{kjsh}\nabla_i M) = 0\end{aligned} \tag{2.14}$$

Multiplying (2.14) by $g^{hk}g^{ij}$, we find that

$$\Big(\frac{M^2}{n^2} + \frac{c}{n}\Big)(2B_s + 2D_s + nA_s) - \frac{(n+2)}{n^2}\nabla_s M^2 = 0 \tag{2.15}$$

Similarly, multiplying (2.14) by $g^{ik}g^{hs}$, we can easily obtain that

$$\Big(\frac{M^2}{n^2}+\frac{c}{n}\Big)(B_s+A_s+(n-1)D_s)-\frac{(n+2)}{2n^2}\nabla_s M^2=0 \tag{2.16}$$

Using (2.15) and (2.16), we get

$$M^2=-cn \qquad \text{or} \qquad A_s=2D_s \tag{2.17}$$

On the other hand, from (1.4), we have

$$\nabla_l Z_{hijk}=A_l Z_{hijk}+B_h Z_{lijk}+B_i Z_{hljk}+D_j Z_{hilk}+D_k Z_{hijl} \tag{2.18}$$

Permutating j,k and l by cyclic in (2.18), adding the three equations and using the expression (1.5) and the first Bianchi Identity, we obtain

$$\begin{aligned}&(A_l-2D_l)Z_{hijk}+(A_j-2D_j)Z_{hikl}+(A_k-2D_k)Z_{hilj}\\&-\frac{1}{n(n-1)}(G_{hijk}\nabla_l R+G_{hikl}\nabla_j R+G_{hilj}\nabla_k R)\end{aligned} \tag{2.19}$$

Transvecting (2.19) with $g^{ij}g^{hk}$, we can obtain

$$2(A_k-2D_k)g^{hk}G_{hl}=\frac{(n-2)}{n}\nabla_l R \tag{2.20}$$

If $A_k = 2D_k$, from (2.20), then we say that the scalar curvature of this hypersurface is constant. If $A_k \neq 2D_k$, from (2.17), the mean curvature of this hypersurface must be constant. If $c=0$ then it is clear that this hypersurface is totally geodesic. Thus, the proof is completed. □

Theorem 2.3 *If a totally geodesic hypersurface of a $(WZS)_{n+1}$ satisfies the condition $R=(1-\frac{2}{n+1})\bar{R}$ then this hypersurface is $(WZS)_n$.*

Proof From (1.4) and (2.4), the proof is easily seen that.

3 Totally umbilical hypersurface of a pseudo concircular symmetric manifold

We consider a non-concircular flat Riemannian manifold (M,g) whose concircular curvature tensor Z_{hijk} satisfies the condition

$$\nabla_l Z_{hijk}=2\lambda_l Z_{hijk}+\lambda_h Z_{lijk}+\lambda_i Z_{hljk}+\lambda_j Z_{hilk}+\lambda_k Z_{hijl} \tag{3.1}$$

where λ_l is a non-zero covariant vector. Such a manifold will be called a pseudo-concircular symmetric manifold and denoted by $(PZS)_n$. Permutating j,k,l by cyclic in (3.1), we obtain the following equations

$$\nabla_j Z_{hikl}=2\lambda_j Z_{hikl}+\lambda_h Z_{jikl}+\lambda_i Z_{hjkl}+\lambda_k Z_{hijl}+\lambda_l Z_{hikj} \tag{3.2}$$

and

$$\nabla_k Z_{hilj} = 2\lambda_k Z_{hilj} + \lambda_h Z_{kilj} + \lambda_i Z_{hklj} + \lambda_l Z_{hikj} + \lambda_j Z_{hilk} \tag{3.3}$$

Adding the equations (3.1), (3.2) and (3.3) and by using the first and the second Bianchi identities, it is obtained that

$$G_{hijk}\nabla_l R + G_{hikl}\nabla_j R + G_{hilj}\nabla_k R = 0 \tag{3.4}$$

Transvecting (3.4) with $g^{hk}g^{ij}$, we get $(1-n)(2-n)\nabla_l R = 0$.

Since $n > 2$, we find that the scalar curvature of the hypersurface is constant. Now, we can state the following theorem:

Theorem 3.1 *The scalar curvature of a pseudo concircular symmetric manifold is constant.*

Theorem 3.2 *Let us suppose that a hypersurface $(PZS)_n$ of a pseudo concircular symmetric manifold $(PZS)_{n+1}$ be totally umbilical. Then the scalar curvature of $(PZS)_{n+1}$ is constant.*

Proof Taking the relation $\frac{A_s}{2} = B_s = D_s = \lambda_s$ in (2.3), (2.4) and (2.5) and using the equation (3.1), we get

$$\begin{aligned}
&\Big(\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\Big)(2\lambda_s G_{hijk} + \lambda_i G_{hsjk} + \lambda_j G_{hisk} + \lambda_k G_{hijs} + \lambda_h G_{sijk}) \\
&\qquad - \frac{1}{n^2} G_{hijk}\nabla_s M^2 - \frac{1}{n} G_{hijk}\nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big) \\
&\quad - \frac{M}{n^2}(G_{hisk}\nabla_j M + G_{ihsj}\nabla_k M + G_{sijk}\nabla_h M + G_{kjsh}\nabla_i M) = 0
\end{aligned} \tag{3.5}$$

Multiplying (3.5) by $g^{hk}g^{ij}$ and $g^{ik}g^{hs}$, respectively, we obtain

$$\begin{aligned}
&\Big(\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\Big)2\lambda_s(2+n) - \frac{(n+2)}{n^2}\nabla_s M^2 \\
&\qquad - \nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big) = 0
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\Big(\frac{M^2}{n^2} + \frac{1}{n}\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big)\Big)\lambda_s(2+n) - \frac{(n+2)}{2n^2}\nabla_s M^2 \\
&\qquad - \frac{1}{n}\nabla_s\Big(\frac{R}{n-1} - \frac{\bar{R}}{n+1}\Big) = 0
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$-\frac{\bar{R}}{n+1} + \frac{R}{n-1} = c \tag{3.8}$$

where c is a positive constant. By using Theorem 3.1, we can say that

$$\bar{R} \equiv \text{const.} \tag{3.9}$$

□

Theorem 3.3 *If a totally geodesic hypersurface of $(PZS)_{n+1}$ satisfies the condition $R = (1 - \frac{2}{n+1})\bar{R}$ then the hypersurface is $(PZS)_n$.*

Proof Let us suppose that a hypersurface of $(PZS)_{n+1}$ be totally geodesic. From the expressions (1.12) and (2.4) and the condition $\frac{A_s}{2} = B_s = D_s = \lambda_s$, the proof is clear. □

4 An example of a $(WZS)_n$

In this section, we want to construct a $(WZS)_n$ spaces. On the coordinate space R^n (with coordinates $x^1, x^2, \ldots, x^n$), we define a Riemannian space V^n and calculate the components of the curvature tensor and its covariant derivative.

Let each Latin index run over $1, 2, \ldots, n$ and each Greek index over $2, 3, \ldots,$ $n-1$. We define a Riemannian metric on R^n $(n > 3)$ by the formula

$$ds^2 = \phi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n \tag{4.1}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants and ϕ is a function of $(x^1, x^2, \ldots, x^{n-1})$ and independent of x^n. In the metric considered, the only non-vanishing components of the curvature tensor, [9]

$$R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta} \tag{4.2}$$

where "." denotes the partial differentiation with respect to the coordinates and $k^{\alpha\beta}$ are the elements of the matrix inverse to $[k^{\alpha\beta}]$.

We consider V_n and

$$\phi = f(x^1)(V_{\alpha\beta}x^\alpha x^\beta \cos g(x^1) + w_{\alpha\beta}x^\alpha x^\beta \sin g(x^1) + k_{\alpha\beta}x^\alpha x^\beta h(x^1))$$

where f, g, h are functions of x^1 only and the matrices $[w_{\alpha\beta}]$, $[V_{\alpha\beta}]$ and $[k_{\alpha\beta}]$ are the form

$$w_{\alpha\beta} = -1 \;\text{ for } \alpha = \beta \qquad \text{and} \qquad w_{\alpha\beta} = 0 \;\text{ for } \alpha \neq \beta \tag{4.3}$$

$$V_{\alpha\beta} = 1 \;\text{ for } \alpha = \beta \qquad \text{and} \qquad V_{\alpha\beta} = 0 \;\text{ for } \alpha \neq \beta \tag{4.4}$$

and

$$k_{\alpha\beta} = \left\{ \begin{matrix} 1 & \text{for } \alpha = \beta \\ 0 & \text{otherwise} \end{matrix} \right\} \tag{4.5}$$

From (4.2), the only non-vanishing components of the concircular curvature tensor Z_{hijk} are

$$Z_{1\alpha\beta 1} = \left\{ \begin{matrix} f(\cos g - \sin g + h) & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{matrix} \right\} \tag{4.6}$$

Here, we consider

$$A_i = B_i = D_i = 0 \text{ for } i \neq 1 \text{ and } A_1 + B_1 + D_1 = c_1,\ c_1 \neq 0 \text{ and const.} \tag{4.7}$$

Thus, from (1.4), V_n will be $(WZS)_n$ if and only if the following relations

$$\nabla_1 Z_{1\alpha\alpha 1} = A_1 Z_{1\alpha\alpha 1} + B_1 Z_{1\alpha\alpha 1} + B_\alpha Z_{11\alpha 1} + D_\alpha Z_{1\alpha 11} + D_1 Z_{1\alpha\alpha 1} \tag{4.8}$$

$$\nabla_\alpha Z_{11\alpha 1} = A_\alpha Z_{11\alpha 1} + B_1 Z_{\alpha 1\alpha 1} + B_1 Z_{1\alpha\alpha 1} + D_\alpha Z_{11\alpha 1} + D_1 Z_{11\alpha\alpha} \tag{4.9}$$

$$\nabla_\alpha Z_{1\alpha 11} = A_\alpha Z_{1\alpha 11} + B_1 Z_{\alpha\alpha 11} + B_\alpha Z_{1\alpha 11} + D_1 Z_{1\alpha\alpha 1} + D_1 Z_{1\alpha 1\alpha} \tag{4.10}$$

Thus, using (4.8), (4.9) and (4.10), we find

$$\begin{aligned} f'(x^1)(\cos g - \sin g + h) + f(x^1)(-g'\sin g - g'\cos g + h') \\ = (A_1 + B_1 + D_1) f(x^1)(\cos g - \sin g + h). \end{aligned} \tag{4.11}$$

By the aid of (4.11), we get

$$f(\cos g - \sin g + h) = c_2 e^{(A_1+B_1+D_1)x^1}, \qquad c_2 > 0. \tag{4.12}$$

So, the n-dimensional weakly concircular recurrent Riemannian manifold has the metric of the form

$$ds^2 = \phi (dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n,$$
$$\phi = c_2 e^{c_1 x^1} \sum_{k=2}^{n-1} (x^k)^2.$$

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