Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **48** (2009) 157–170

Metrizability of Connections on Two-Manifolds*

Alena VANŽUROVÁ [1], Petra ŽÁČKOVÁ [2]

[1] *Department of Algebra and Geometry, Faculty of Science, Palacký University*
tř. 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: alena.vanzurova@upol.cz

[2] *Department of Mathematics and Didactics of Mathematics*
Technical University Liberec, Studentská 2, 46117 Liberec, Czech Republic
e-mail: petrazacek@centrum.cz

(Received May 6, 2009)

Abstract

We contribute to the reverse of the Fundamental Theorem of Riemannian geometry: if a symmetric linear connection on a manifold is given, find non-degenerate metrics compatible with the connection (locally or globally) if there are any. The problem is not easy in general. For nowhere flat 2-manifolds, we formulate necessary and sufficient metrizability conditions. In the favourable case, we describe all compatible metrics in terms of the Ricci tensor. We propose an application in the calculus of variations.

Key words: Manifold, linear connection, metric connection, pseudo-Riemannian geometry.

2000 Mathematics Subject Classification: 53B05, 53B20

1 Preliminaries—affine differential geometry

Recall briefly some well-known facts from affine and metric differential geometry. Let M be an n-dimensional smooth manifold ("smooth" always means of the class C^∞), T_xM the tangent space at $x \in M$, and let $\pi\colon TM \to M$ denote the tangent vector bundle of M. $\mathcal{F}(M) = C^\infty(M)$ denotes the ring of all smooth functions on M, $\mathcal{X}(M)$ the $C^\infty(M)$-module of all smooth vector fields on M (which can be viewed as sections of the projection π), and $\Lambda(M)$ the exterior algebra over M. $\pi^1\colon J^1TM \to M$ is the first jet prolongation of the tangent

*Supported by the Research and Development Council of the Czech Government MSM 6 198 959 214.

vector bundle $\pi\colon TM \to M$, that is, the fibred manifold of 1-jets in $J^1(M, TM)$ which may be represented by local sections of the projection π. We have also a canonical projection $\pi^1_0\colon J^1TM \to TM = J^0TM$. Given an n-dimensional smooth manifold M, a (generalized) *connection*[1] on TM is a (smooth) section $\Gamma\colon TM \to J^1TM$ of π^1_0. A section Γ of π^1_0 which is linear as a fibred morphism of vector bundles is called a *linear connection* on TM, [10], [8]. Any linear connection Γ on TM induces the so-called covariant derivative on M, and vice versa. Recall that a *covariant derivative* on M is a mapping $(X, Y) \mapsto \nabla_X Y$, $\nabla\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, such that

$$\begin{gathered}\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \qquad \nabla_X(fY) = f\nabla_X Y + (Xf)Y,\\ \nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z\end{gathered} \tag{1}$$

for any vector fields X, Y, Z on M and functions $f, g \in \mathcal{F}(M)$ on M; often, under a linear connection on M we mean just ∇. To emphasise that ∇ arises from a linear connection Γ we can write ∇^Γ. In what follows, (M, ∇) will denote a manifold with linear[2] connection in the above sense.

If (U, φ), $U \subset M$ open, $\varphi = (x^1, \dots, x^n)$ is a local chart on M denote by (x^i, v^i) the induced adapted coordinates on $V = \pi^{-1}(U) \subset TM$ and by (x^i, v^i, v^i_j) the corresponding fibre coordinates on $(\pi^1_0)^{-1}(V) \subset J^1TM$. A connection Γ on TM can be locally given by functions $v^i_j \circ \Gamma = \Gamma^i_j(x, v)$ called *components* of Γ. A connection is linear if and only if its components are just linear functions in v^k, that is, there exist functions Γ^i_{jk} of coordinates on $U \subset M$ such that $\Gamma^i_j(x, v) = \Gamma^i_{jk}(x)v^k$ holds.

If (x^i) are local coordinates on $U \subset M$, we can introduce *components* (Christoffel symbols) of ∇ relative to the chart under consideration directly as the functions $\Gamma^k_{ij}(x)$ given on U by[3] $\nabla_i \frac{\partial}{\partial x^j} := \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$. Note that the linear connection Γ (or ∇, respectively) is fully determined by components Γ^k_{ij} provided they satisfy the well-known transformation law on overlappings of neighborhoods, [9, I, Ch. 3, Th. 7.2, Th. 7.3]; recall that Γ^k_{ij} are not components of a tensor.

Covariant derivation extends to tensor fields, [9, I]: if F is of type (r, s) then $\nabla_X F$ is of the same type, and ∇F is of type $(r, s+1)$.

The *torsion* of a manifold (M, ∇) with linear connection is a type $(0, 2)$ tensor field T given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ for $X, Y \in \mathcal{X}(M)$. Here $[\,,\,]$ is the Lie bracket, $[X, Y]f = X(Yf) - Y(Xf)$ for $f \in \mathcal{F}(M)$; T is skew-symmetric. The *curvature* of (M, ∇) is a type $(0, 3)$ tensor field R defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$.

The map $R(X_x, Y_x)\colon T_xM \to T_xM$ is linear and skew-symmetric, $R(Y, X) = -R(X, Y)$. A connection ∇ is called *torsion-free* (*torsion-less*, or *symmetric*) if

[1] In the sense of Ehresmann

[2] Many authors still use the term “affine connection” instead, from historical reasons; note that affine connection or affine manifold may have a different meaning: each tangent space T_xM is considered as an affine space, and $TM \to M$ as an affine bundle, similarly for morphisms etc., [9, I, Ch. 3].

[3] As usually, $\langle \frac{d}{dx^1}, \dots, \frac{d}{dx^n} \rangle$ is a basis of coordinate vector fields.

$T \equiv 0$ (in local coordinates, $\Gamma^i_{jk} = \Gamma^i_{kj}$), and *flat* if $T \equiv 0$ and $R \equiv 0$. ∇ is flat if and only if around any point, there are local coordinates such that $\Gamma^i_{jk} = 0$ holds. We introduce the *Ricci tensor* Ric of type $(0,2)$ as a trace of a linear map, namely $\mathrm{Ric}(Y,Z) = \mathrm{Tr}\{X \mapsto R(X,Y)Z\}$ (the other possibility differs up to a sign). Components of torsion $T = T^i_{jk}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k$, of curvature $R = R^i_{hjk}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^h$ and of Ricci tensor $\mathrm{Ric} = R_{jk}dx^j \otimes dx^k$ in terms of components of connection are $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{jk}$,

$$R^i_{hjk} = \frac{\partial \Gamma^i_{kh}}{\partial x^j} - \frac{\partial \Gamma^i_{jh}}{\partial x^k} + \sum_s \left(\Gamma^i_{js}\Gamma^s_{kh} - \Gamma^i_{ks}\Gamma^s_{jh}\right), \tag{2}$$

$$R_{jk} = \sum\nolimits_i R^i_{kij} = \sum_i \left(\frac{\partial \Gamma^i_{jk}}{\partial x^i} - \frac{\partial \Gamma^i_{ik}}{\partial x^j}\right) + \sum_{i,s} \left(\Gamma^i_{is}\Gamma^s_{jk} - \Gamma^i_{js}\Gamma^s_{ik}\right). \tag{3}$$

Due to the co-called first Bianchi Identity ($R^i_{[hjk]} = 0$)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \tag{4}$$

and antisymmetry of the curvature we get

$$R_{jk} - R_{kj} = \sum_i \left(R^i_{kij} + R^i_{jki}\right) = R^i_{ikj} = \mathrm{Tr}\ \mathrm{R}_{kj} = \sum_s \frac{\partial \Gamma^s_{sj}}{\partial x^k} - \frac{\partial \Gamma^s_{sk}}{\partial x^j}. \tag{5}$$

Hence in general, the Ricci tensor is not necessarily symmetric, even for a symmetric connection. We can see the following:

Lemma 1 *The Ricci tensor satisfies [14, p. 14]*

$$\mathrm{Ric}(Z,Y) - \mathrm{Ric}(Y,Z) = \mathrm{Tr}\, R(Y,Z).$$

In general, the functions $\psi_i = \sum_s \Gamma^s_{is}$ ("traces") that appear in (5) do not transform as components of a tensor (1-form) since Γ^i_{jk} do not, either. Nevertheless, they play the following role:

Lemma 2 (Local necessary and sufficient condition for symmetry of Ric) *The following conditions are equivalent for* (M,∇)*:*
(i) The Ricci tensor Ric *is symmetric on* M*.*
(ii) The curvature tensor R *is trace-less,* $\mathrm{Tr}\, R = 0$*.*
(iii) In each coordinate neighborhood the components of connection satisfy

$$\frac{\partial \Gamma^s_{is}}{\partial x^j} - \frac{\partial \Gamma^s_{js}}{\partial x^i} = 0, \qquad i,j = 1,\ldots,n. \tag{6}$$

The equations (6) in fact tell that there is a function f^U on U such that $\psi_i = \sum_s \Gamma^s_{is} = \frac{df^U}{dx^i}$, $i = 1,\ldots,n$; ψ_i is a "gradient vector". That is, if we introduce a one-form on a coordinate nbd U by $\psi^U = \sum_i \psi_i dx_i = \Gamma^s_{is}dx^i$ then (6) is a necessary and sufficient condition for ψ^U be closed on U, $d\psi^U = 0$.

Recall that an *exterior q-form* ω on M is a totally antisymmetric type $(0,q)$ field; ω is *closed* if $d\omega = 0$, and *exact* if $\omega = d\alpha$ for some $(q-1)$-form α. Since $d^2 = 0$, exact forms are obviously closed, but not vice versa. The so-called Poincaré lemma guarantees that any closed form is locally exact. Obviously, a form α from the above formula is not determined by ω uniquely (in fact, there are many $(q-1)$-forms with the same differential).

Symmetry of the Ricci tensor is closely related to the concept of parallel volume element. We say that (M, ∇), $\dim M = n$, is *locally equiaffine*, or *volume preserving* if locally, around each point $x \in M$, there exists a non-vanishing and covariantly constant n-form ω; $\nabla\omega = 0$. If this is the case, ω is called a (local) *volume element*. The following holds, [14]:

Lemma 3 *(M, ∇) with $T \equiv 0$ is locally equiaffine if and only if the Ricci tensor is symmetric.*

(M, ∇) with $T \equiv 0$ is called *equiaffine* if it admits a parallel volume element. If M is simply connected and (M, ∇) is locally equiaffine then it is equiaffine [14, p. 15]. Hence a symmetric linear connection with a trace-less curvature tensor (equivalently, with symmetric Ric) on a simply connected manifold is equiaffine.

1.1 Parallelism and recurrency

If $c\colon I \to M$, $t \mapsto c(t)$ is a curve, let $\zeta(t) = (c(t), c'(t))$ denote the corresponding tangent vector field along the curve c; $c'(t) = \frac{dc}{dt}$. Let Y be a vector field along c. Then the covariant derivative $\nabla_\zeta Y$ along c is defined; in terms of local coordinates, if $Y = Y^k(t)(\frac{\partial}{\partial x^k})_{c(t)}$ then

$$\nabla_\zeta Y = \sum_k \left(\frac{dY^k}{dt} + \sum_{i,j} \Gamma^k_{ij}(c(t)) \frac{dc^i}{dt} Y^j \right) \frac{\partial}{\partial x^k}.$$

A regular[4] differentiable curve $t \mapsto c(t)$ is an *unparametrized geodesic*[5], [13], or *pregeodesic*, [14], if there is a real function $\phi(t)\colon I \to \mathbb{R}$ along c such that $\nabla_\zeta \zeta = \phi\zeta$. Equations of (pre)geodesics read $x''^i + \Gamma^i_{jk} x'^j x'^k = \phi\, x'^i$. If the tangent vector field is parallel along the curve, $\nabla_\zeta\zeta = 0$, we speak on *canonically parametrized geodesics*; the so-called *canonical affine parameter* s is determined uniquely up to affine transformations $s \mapsto as+b$ with $a \neq 0$. In local coordinates, canonically parametrized geodesics are described by the well-known system of differential equations

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \qquad j,k = 1, \dots, n. \tag{7}$$

Connections with the same "symmetric part" ∇_s, $\nabla_s(X,Y) = \frac{1}{2}(\nabla(X,Y) + \nabla(Y,X))$, have the same geodesics, and pregeodesics, too.

[4] in the sense that $\dot{c}(s) = \frac{dc}{ds} \neq 0$ for all $s \in I$

[5] to emphasize that the particular parametrization is unrelevant for actual considerations

A diffeomorphism $f\colon (M,\nabla) \to (\hat{M},\hat{\nabla})$ is called a *geodesic mapping* if all geodesics of (M,∇) are mapped into unparametrized geodesics of $(\hat{M},\hat{\nabla})$.

A non-vanishing tensor field F on (M,∇) is called *parallel*, or *covariantly constant* (with respect to ∇) if $\nabla F = 0$; equivalently[6], $\nabla_X F = 0$ for any $X \in \mathcal{X}(M)$. A non-vanishing tensor field F on M is *recurrent* if there is a one-form ω such that

$$\nabla F = \omega \otimes F. \tag{8}$$

Lemma 4 *Let a type (r,s) tensor field F on (M,∇) be recurrent; $\nabla F = \omega \otimes F$ for some 1-form. Let F be non-vanishing on M. Then the 1-form ω is closed.*

Proof Recurrency means that for arbitrary vector fields $Y_1,\dots,Y_s$ and one-forms $\omega^1,\dots,\omega^r$ on M,

$$(\nabla_X F)(Y_1,\dots,Y_s,\omega^1,\dots,\omega^r) = \omega(X)\cdot F(Y_1,\dots,Y_s,\omega^1,\dots,\omega^r).$$

In local coordinates about any point $p \in M$, let $\omega = \omega_k\, dx^k$, and $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. It follows that $\nabla_k F^{i_1\dots i_r}_{j_1\dots j_s} = \omega_k \cdot F^{i_1\dots i_r}_{j_1\dots j_s}$ for any $k = 1,\dots,n$; $n = \dim M$. Let the component $F^{i_1\dots i_r}_{j_1\dots j_s}$ (for fixed indices) be non-zero at p, and due to continuity, in some nbd U of p (from continuity again, the component is either positive, or negative around the point). Then the components of the 1-form can be expressed in U as

$$\omega_k = \frac{1}{F^{i_1\dots i_r}_{j_1\dots j_s}}\cdot \nabla_k F^{i_1\dots i_r}_{j_1\dots j_s} = \nabla_k\,(\ln|F^{i_1\dots i_r}_{j_1\dots j_s}|) = \frac{\partial}{\partial x^k}(\ln|F^{i_1\dots i_r}_{j_1\dots j_s}|),\ k = 1,\dots,n.$$

That is, about any point $p \in M$, $\omega = d(\ln|F^{i_1\dots i_r}_{j_1\dots j_s}|)$; i.e. ω is locally exact, and $d\omega = d(df) = 0$. □

Lemma 5 *Let F be a type (r,s) tensor field on (M,∇). Let $\alpha \in \mathcal{F}(M)$ be a non-vanishing real function; $\alpha(x) \neq 0$ for $x \in M$. Then the following conditions are equivalent:*

- *$\alpha \otimes F$ is parallel with respect to ∇,*
- *$\nabla F = d(-\ln|\alpha|) \otimes F$.*

Proof Since $\nabla(\alpha\otimes F) = (\nabla\alpha)\otimes F + \alpha\otimes(\nabla F)$ and $\alpha \neq 0$, we have: $\nabla(\alpha\otimes F) = 0$ iff $\nabla F = -(\frac{1}{\alpha}\cdot\nabla\alpha)\otimes F = -d(\ln|\alpha|)\otimes F$. Hence $\alpha\otimes F$ is parallel if and only if $\nabla F = df \otimes F$ where $f = -\ln|\alpha|$. □

Lemma 6 *If a tensor field F of type (r,s) on (M,∇) is recurrent, $\nabla F = \omega\otimes F$, and the 1-form is exact, $\omega = df$, then $e^{-f}\otimes F$ is parallel w.r.t. ∇.*

Proof If $\nabla F = df \otimes F$ denote $\alpha = e^{-f}$. Then $f = -\ln\alpha$, and $\nabla(\alpha\otimes F) = d\alpha\otimes F + \alpha\cdot d(-\ln\alpha)\otimes F = d\alpha\otimes F + \alpha\cdot(-\frac{1}{\alpha})\cdot d\alpha\otimes F = 0$. Hence $\alpha\otimes F = e^{-f}\otimes F$ is parallel. □

[6] In more geometric language, the condition tells that the field is preserved under parallel transport along all curves in M.

1.2 Compatible metrics

Recall that a *pseudo-Riemannian metric* on a smooth manifold M is a (smooth) type $(0,2)$ tensor field on M such that in any point $x \in M$, the corresponding bilinear form g_x defined on T_xM is symmetric and non-degenerate; (M,g) is called a *pseudo-Riemannian* manifold. If g_x is moreover positive definite for all $x \in M$, (M,g) is called the *Riemannian space*. A linear connection ∇ (may be non-symmetric in general) on (M,g) is *compatible* with g if g is parallel with respect to ∇, $\nabla g = 0$.

The Fundamental Theorem of Riemannian geometry states that any pseudo-Riemannian manifold (M,g) admits a unique linear connection $\tilde{\nabla}$, called the Riemannian (or Levi-Civita) connection, or metric connection, of (M,g), characterized by the pair of conditions $T \equiv 0$, $\tilde{\nabla} g = 0$ (the parallel transport with respect to $\tilde{\nabla}$ along any curve preserves the scalar product of tangent vectors defined by g). On (M,g), components Γ^i_{jk} of the Levi-Civita connection are related to components g_{ij} of the metric by the well-known formula $\Gamma^\ell_{ik} = \frac{1}{2} g^{\ell j} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j} \right)$.

On the other hand, given a manifold equipped with a linear connection, (M,∇), we might be interested in metrics the given connection is compatible with. If ∇ is torsion-free, it means to find a metric g on M such that ∇ is just the Levi-Civita connection of (M,g). We say that a manifold (M,∇) is *metrizable*, or *locally metrizable*, respectively, if there exists a metric (or exists locally, respectively) compatible with the connection (metrization problem, MP).

Essentially the same problem can be formulated in a bit more general setting as follows, [13] (the answer is formulated in Corollary 1): *If* (M,∇) *is given find all geodesic mappings (i.e. diffeomorphisms which map geodesics onto unparametrized geodesics) of* (M,∇) *onto (all possible) pseudo-Riemannian manifolds* $(\bar{M},g)$ *(due to diffeomorphisms, we can in fact suppose* $\bar{M} = M$*).*

In local coordinates, the formula $\nabla g = 0$ reads[7]

$$\frac{\partial g_{ij}}{\partial x^k} = g_{sj}\Gamma^s_{ik} + g_{is}\Gamma^s_{jk}. \tag{9}$$

In principle, to answer the question on (local) metrizability of a connection means to solve the system[8] (9). Employing the curvature, necessary integrability conditions for metrizability can be given in the form of an infinite system of linear equations in $\frac{1}{2}n(n+1)$ functions g_{ij} (with coefficients which are functions in $\Gamma' s$ and their partial derivatives), [7]; the coordinate-free form reads

$$g(R(X,Y)Z,W) + g(Z,R(X,Y)W) = 0, \tag{10}$$

$$g(\nabla^r R(X,Y;Z_1;\ldots;Z_r)(Z),W) + g(Z,\nabla^r R(X,Y;Z_1;\ldots;Z_r)(W)) = 0 \tag{11}$$

for all $X,Y,Z,W,Z_1,\ldots,Z_r \in \mathcal{X}(M)$, $1 \le r < \infty$. Flat connections are locally metrizable[9]. If (10) has at least a 1-dimensional solution space containing a

---

[7] In components, $g_{ij;k} =: \nabla g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}; \frac{\partial}{\partial x^k}) = \frac{\partial g_{ij}}{\partial x^k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk}$.

[8] Which can be done directly in simple cases.

[9] For the detailed theory of flat affine manifolds, cf. [9, I], flat Riemannian manifolds are discussed e.g. in [23].

non-degenerate metric and any solution of (10) satisfies also (11) for $r = 1$ then (M, ∇) is metrizable, [7].

Corollary 1 *If there exist $\frac{1}{2}n(n+1)$ (differentiable) functions g_{ij} which solve the system*

$$g_{sj} R^s_{ik\ell} + g_{is} R^s_{jk\ell} = 0 \tag{12}$$

and satisfy $g_{ij} = g_{ji}$, $\det(g_{ij}) \neq 0$, and any solution of (12) solves the system

$$g_{sj} R^s_{ik\ell;m} + g_{is} R^s_{jk\ell;m} = 0 \tag{13}$$

then (locally) there exist geodesic mappings of (M, ∇) onto pseudo-Riemannian spaces.

On a (pseudo-)Riemannian manifold (M, g) with the metric tensor g besides the curvature tensor R in type $(1,3)$, we can consider the type $(0,4)$ tensor $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, usually also called curvature tensor; the relations $\tilde{R}(X,Y,Z,W) = \tilde{R}(Z,W,X,Y) = -\tilde{R}(Y,X,Z,W) = -\tilde{R}(X,Y,W,Z)$ hold. In a coordinate system $(U, \varphi = (x^i))$ based at a point $x \in M$, components R^ℓ_{ijk} of R and R_{hijk} of $\tilde{R} = R_{hijk} dx^j \otimes dx^k \otimes dx^i \otimes dx^h$ are related by $R_{hijk} = g_{hs} R^s_{ijk}$, and $g^{\ell h} R_{hijk} = R^\ell_{ijk}$[10].

Lemma 7 *The Ricci tensor of the Levi-Civita connection of a (pseudo-)Riemannian manifold (M, g) is always symmetric, [6, p. 331].*

The sectional curvature of a two-space P given by the linearly independent tangent vectors $X, Y \in T_xM$ is given by

$$K(X \wedge Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{\tilde{R}(X,Y,Y,X)}{||X \wedge Y||^2} \tag{14}$$

where $||X \wedge Y||$ is the area of a parallelogram determined by X and Y, [3, p. 94], [6, p. 327] etc. The sectional curvature determines the whole curvature tensor $\tilde{R}$, [8, p. 137].

On (M, g), the Ricci tensor in type $(1,1)$ is introduced with components $R^i_j = g^i_s R_{sj}$, and the scalar curvature ϱ as its trace, $\varrho = \operatorname{Tr} \operatorname{Ric} = R^s_s = g^{ij} R_{ij}$.

A Riemannian manifold (M, g) is called *isotropic at a point* $x \in M$ if the curvature is the same constant, $K(x)$, on every (two-plane) section, and *isotropic* if it is isotropic at every point, [1]. If x is an isotropic point of (M, g) then the following formula holds at x in any local coordinates around x:

$$R_{hijk} = K(x)(g_{hj} g_{ik} - g_{hi} g_{jk}). \tag{15}$$

A two-dimensional manifold is (trivially) isotropic, therefore it satisfies (15).

Pseudo-Riemannian manifolds with symmetric Ricci tensor for which the Ricci tensor is proportional to the metric tensor, $\operatorname{Ric} = \lambda g$, are called *Einstein spaces*, [12, p. 263], [15], [17]. In the Loretzian case, they are important in

[10] As already mentioned, $R_{hijk} = R_{jkhi} = -R_{ihjk} = -R_{hikj}$.

Einstein's theory of general relativity (the Einstein's field equation is a dynamical equation which describes how matter changes the geometry of spacetime; in vacuum, it is given by the condition $\mathrm{Ric} = 0$). The factor of proportionality can be calculated[11], $\lambda = \frac{1}{n}\varrho$, hence for Einstein spaces,

$$\mathrm{Ric} = \frac{1}{n}\varrho g. \tag{16}$$

Particularly, all two-dimensional pseudo-Riemannian manifolds are Einstein spaces as we check below, cf. [12, p. 263], [15, p. 101].

2 Metrizability of 2-manifolds

Let us pay attention to existence of compatible metrics in the simplest case $n = \dim M = 2$. Let (x^1, x^2) denote local coordinates on a coordinate neighborhood U of a manifold M_2. In dimension two, the curvature is simply given by $R_{hijk} = K(x)(g_{hj}g_{ik} - g_{hi}g_{jk})$ [8, p. 137], and the function $K(x)$ is called the *Gauss curvature.* The Riemann curvature R in type $(1,3)$ and the Ricci tensor Ric are related by [12], [15]

$$R^i_{hjk} = \delta^i_j R_{kh} - \delta^i_k R_{jh}. \tag{17}$$

As far as $R^i_{hjj} = 0$ and $R^i_{hij} = R_{jh}$ holds for $j \neq i$, the curvature tensor of a linear connection ∇ on M_2 is completely determined by its Ricci tensor; explicitely,

$$\begin{aligned} R_{11} &= -R^2_{112} = R^2_{121}, \quad & R_{21} &= -R^1_{121} = R^1_{112}, \\ R_{12} &= -R^2_{212} = R^2_{221}, \quad & R_{22} &= -R^1_{221} = R^1_{212}. \end{aligned} \tag{18}$$

Particularly, $R = 0$ if and only if $\mathrm{Ric} = 0$, and recurrency is also inherited:

Lemma 8 *For (M_2, ∇),* Ric *is recurrent if and only if R is recurrent.*

Proof Let Ric be recurrent, $\nabla\mathrm{Ric} = \omega \otimes \mathrm{Ric}$. In local coordinates, if $\omega = \omega_j dx^j$ then $\nabla_\ell R^i_{hjk} = \delta^i_j \nabla_\ell R_{kh} - \delta^i_k \nabla_\ell R_{jh} = \delta^i_j \omega_\ell R_{kh} - \delta^i_k \omega_\ell R_{jh} = \omega_\ell R^i_{hjk}$, hence $\nabla R = \omega \otimes R$. Vice versa, if $\nabla R = \omega \otimes R$ holds then $\nabla_\ell R_{jk} = \omega_\ell R^i_{kij} = \omega_\ell R_{jk}$, and $\nabla\mathrm{Ric} = \omega \otimes \mathrm{Ric}$. □

On (M_2, g), non-zero components of type $(0,4)$ curvature $\tilde{R}$ are (up to a sign) equal just R_{1212}, and (15) reads ([15, p. 62], [8, p. 137])

$$R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{19}$$

where $K = K(x)$ is the Gauss curvature, $K = \frac{R_{1212}}{\det(g_{ij})}$.

Lemma 9 *The curvature tensor of a two-dimensional pseudo-Riemannian manifold (M_2, g) satisfies*

$$R^i_{hjk} = K(\delta^i_k g_{hj} - \delta^i_j g_{hk}), \tag{20}$$

and the Ricci tensor is proportional to the metric tensor,

$$\mathrm{Ric} = K \cdot g = \frac{1}{2}\varrho \cdot g. \tag{21}$$

---

[11] In fact, $\varrho = R_{ij}g^{ij} = \lambda g_{ij}g^{ij} = n\lambda$.

Proof We can either use the fact that M_2 is trivially isotropic, [1, p. 374], and (16) holds, or proceed by direct evaluation: $R^t_{hij} = R^s_{hij}\delta^t_s = R^s_{hij}g_{sk}g^{kt} = R_{hijk}g^{kt} = K(g_{hj}g_{ik} - g_{hk}g_{ij})g^{kt} = K(\delta^t_i g_{hj} - \delta^t_h g_{ij})$. It follows immediately for the Ricci tensor that $R_{hj} = \Sigma_i R^i_{hij} = K \cdot \Sigma_i(\delta^i_i g_{hj} - \delta^i_h g_{ij}) = K \cdot g_{hj}$, hence $\mathrm{Ric} = Kg$, and $\varrho = R_{hj}g^{hj} = 2K$. □

Corollary 2 *(M_2, g) is always an Einstein space. For a nowhere flat (M_2, g), the Ricci tensor is symmetric and non-degenerate.*

Note that according to [9, I, p. 280], any non-flat Riemannian 2-manifold has a recurrent curvature provided its sectional curvature does not vanish. We can check:

Lemma 10 *The Ricci tensor of a nowhere flat pseudo-Riemannian manifold (M_2, g) is recurrent, and the corresponding 1-form is exact*[12].

Proof $R \neq 0$ is equivalent with $K(x) \neq 0$ on M (from continuity, K is either positive, or negative). Since by (21), $g = \alpha(x) \cdot \mathrm{Ric}$ with $\alpha(x) = \frac{1}{K(x)} \neq 0$, and $\nabla g = 0$, we get easily that $\alpha(x) \cdot \mathrm{Ric}$ is parallel. According to Lemma 5, $\nabla\mathrm{Ric} = d(-\ln|\alpha|) \otimes \mathrm{Ric}$ holds. □

It follows from the above discussion on pseudo-Riemannian manifolds that two conditions are necessary for local metrizability of a (symmetric) connection on a 2-manifold: the Ricci tensor must be symmetric, and must be also recurrent, with the corresponding 1-form being closed; Ric may be degenerate only in the case $R = 0$, and then $\mathrm{Ric} = 0$ holds. Furthermore, for global metrizability, the 1-form from the recurrency condition must be even exact. A flat connection is always (globally) metrizable, with $\frac{1}{2}n(n+1)$-parameter solution space; even the signature can be prescribed. So let us pay attention to the situation when the curvature tensor (or equivalently, the Ricci tensor) is non-zero in one point $x_0 \in M$, and due to continuity, in some neighborhood of x_0[13].

Theorem 1 (Existence of local metrics on two-manifolds) *Let a 2-dimensional manifold (M_2, ∇) with a symmetric linear connection be given such that the Ricci tensor is regular, $|R_{ij}| \neq 0$, symmetric, $R_{ij} = R_{ji}$, and recurrent, $\nabla Ric = \varrho \otimes Ric$ for some 1-form ϱ. Then locally, there is a metric compatible with the connection.*

Proof Let $x_0 \in M$. $|R_{ij}| \neq 0$ implies existence of a pair (i, j) of indices such that $R_{ij} \neq 0$ about[14] x_0. Recurrency together with regularity guarantee that $d\varrho = 0$ (Lemma 4). Hence about x_0, there is a function f such that $\varrho = df$. Consequently, $e^{-f} \cdot \mathrm{Ric}$ is parallel about x_0. Therefore $g = e^{-f} \cdot \mathrm{Ric}$ is a local metric on a nbd of x_0 compatible with ∇. □

[12] and consequently closed

[13] The subset of non-flat points is open.

[14] Under "about x" we mean on some neighborhood of x.

Of course, the function f from the proof is not unique. Any function $\tilde{f}$ with the same differential, $d\tilde{f} = df$, also gives a metric; such a function differs up to a constant, $\tilde{f} = f + a$, $a \in \mathbb{R}$.

If R is nowhere zero, a similar proof quarantees existence of global metrizability of a nowhere flat affine manifold:

Proposition 1 *Let (M_2, ∇) be a two-dimensional manifold with a symmetric linear connection. If the Ricci tensor of ∇ is regular, symmetric, and recurrent, $\nabla Ric = \varrho \otimes Ric$, and the 1-form ϱ is exact, i.e. $\varrho = df$ for some function $f \in \mathcal{F}(M)$, then $g = e^{-f} \cdot \mathrm{Ric}$ is a (global) metric tensor compatible with ∇.*

Theorem 2 (Global metrizability of no-where flat connections on 2-manifolds) *A nowhere flat symmetric linear connection on M_2 is metrizable if and only if its Ricci tensor is regular, symmetric, recurrent, and the corresponding 1-form is exact. If this is the case, and $\nabla Ric = df \otimes Ric$ holds for some smooth function $f \in \mathcal{F}(M)$, then all global metrics compatible with ∇ form a 1-parameter family described by the formula*

$$g_b = \exp(-f + b) \cdot \mathrm{Ric}, \quad b \in \mathbb{R}, \tag{22}$$

that is, any of them arises from the Ricci tensor as a multiple by a smooth function. Moreover, any two compatible metrics differ up to a scalar multiple.

Proof The main statement has been already proved - the "if" part in Theorem 1 and Proposition 1, and the "only if" part in Corollary 2 and Lemma 10. As to the rest, let $g = e^{-f} \cdot \mathrm{Ric}$, $\tilde{g} = e^{-\tilde{f}} \cdot \mathrm{Ric}$ be two compatible metrics, then $\tilde{f} - f = a$, $\mathrm{Ric} = e^{\tilde{f}} g$, and $g = e^{a}\tilde{g}$. We get $\tilde{g} = e^{-f-a} \cdot \mathrm{Ric}$; i.e. (22) holds. □

As an immediate consequence of Theorem 2 we obtain:

Corollary 3 *Two pseudo-Riemannian metrics g_1, g_2 compatible with the same nowhere flat (symmetric) linear connection on M_2 are homothetic.*

Unicity of g declared in [18, p. 532] must be understood in this way.

For positive-definite metrics, this result is a special case of the Theorem 1 of O. Kowalski from [11, p.131] (recall that two metrics g_1, g_2 on a manifold are called *conformally equivalent* if there is a function κ on M such that $g_2 = \kappa g_1$, [23, p. 99]): *Let g, g' be two Riemann metrics on a smooth manifold M with the same Riemann curvature tensor R. Then g, g' are conformally equivalent on the closure of the set of all regular points of R.*

3 Application in the calculus of variations

Let us mention the relationship of our problem to the Calculus of Variations. The so-called Inverse Problem (IP) of the calculus of variations is: if a system $\ddot{x}^i = f^i(t, x^k, \dot{x}^k)$, $i, k = 1, \ldots, n$ of second order differential equations (SODEs) is given, find—sufficiently differentiable—Lagrangian functions $L(t, x^k, \dot{x}^k)$ and a multiplier matrix $g_{ij}(t, x^k, \dot{x}^k)$ such that

$$g_{ij}(\ddot{x}^i - f^i) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}.$$

Given a system of second order ODEs of a particular type

$$\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \qquad k = 1, \dots, n, \tag{23}$$

that is, second derivatives can be expressed as quadratic forms in first derivatives, we can use the above theory for deciding whether the system (23) is derivable from a Lagrangian. In fact, provided $\det(g_{ij}) \neq 0$, the system (23) is equivalent to the system

$$g_{mi}(\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k) = 0, \quad i, m = 1, \dots, n. \tag{24}$$

Another speaking, MP can be viewed as a particular case of IP, where $f^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$ (that is, f^i are quadratic forms in components of velocities, with coefficients depending only on components of positions) in the particular case when the multipliers are time- and velocities-independent. We can assume that the coefficients in (23), the functions $\Gamma^k_{rs}(x)$, are components of a symmetric linear connection ∇ on some neighborhood $U \subset \mathbb{R}^n$. If ∇ is (locally) metrizable, and $g_{ij}(x)$ (with $\det(g_{ij}(x)) \neq 0$ at any $x \in U$) are components of some non-degenerate metric g compatible with ∇ on U, then (23) and (24) are equivalent, hence the functions $g_{ik}(x)$ can be taken as the desired variational multipliers. One of particular Lagrangians comming from MP (and solving IP) is

$$L = T = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j, \tag{25}$$

the kinetic energy. There might exist multipliers of a more general form $g_{ik}(t, x, \dot{x})$, depending on "time, positions and velocities", which might bring more complicated Lagrangians, [5].

4 Examples

Example 1 ([7, p. 122]) On $\mathbb{R}^2$ with coordinates $x = (x^1, x^2)$, assume the system of ODEs

$$(\ddot{x}^1)^2 + (x^1 - x^2)(\dot{x}^1)^2 = 0, \quad (\ddot{x}^2)^2 + (x^1 - x^2)(\dot{x}^2)^2 = 0. \tag{26}$$

Curves $c(s)\colon I \to \mathbb{R}^2$ (parametrized by arcs length), which are solutions of the system, represent the family of geodesics of a (symmetric) linear connection ∇ with components $\Gamma^1_{11} = \Gamma^2_{22} = x^1 - x^2$, $\Gamma^i_{jk} = 0$ otherwise. We ask if the (torsion-free linear) connection is metrizable, i.e. we wish to find type $(0, 2)$ symmetric tensor field g with $\nabla g = 0$. The corresponding system

$$\begin{aligned}
&\partial_1 g_{11} = (x^1 - x^2)g_{11}, \quad \partial_1 g_{12} = 0, \quad \partial_1 g_{22} = 0,\\
&\partial_2 g_{11} = 0, \quad \partial_2 g_{12} = (x^1 - x^2)g_{12}, \quad \partial_2 g_{22} = (x^1 - x^2)g_{22}
\end{aligned}$$

can be solved directly, but the only solution is trivial, $g_{ij} = 0$ for all i, j. Or, argumentation using the Ricci (or curvature) tensor can be used: $R_{11} = R^2_{121} = 0$, $R_{12} = R^1_{112} = 1$, $R_{21} = R^2_{221} = -1$, $R_{22} = R^1_{212} = 0$, hence the Ricci tensor is not symmetric, our linear connection is not metrizable (even locally).

It appears that in this particular case, the quickest and most comfortable way is to use the criterion from Lemma 2 (iii): we check that $\psi_1 = \psi_2 = x^1 - x^2$, $\partial_1\psi_2 = 1$ while $\partial_2\psi_1 = -1$.

Example 2 The system of equations

$$\ddot{x}^1 = -(\dot{x}^1)^2 - (\dot{x}^2)^2, \quad \ddot{x}^2 = -4\dot{x}^1\dot{x}^2 \tag{27}$$

corresponds to a torsion-free linear connection on $\mathbb{R}^2$ with components

$$\Gamma^1_{11} = \Gamma^1_{22} = 1, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^2_{11} = \Gamma^2_{22} = 0, \quad \Gamma^2_{12} = \Gamma^2_{21} = 2.$$

Now our "quick" criterion fails, the connection determined by (27) has symmetric Ricci tensor: $\psi_1 = 3$, $\psi_2 = 0$, $\mathrm{Ric} = (R_{hk}) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$. But the connection is not metrizable, either, since Ricci is not recurrent: system of linear equations for functions $\alpha_1(x)$, $\alpha_2(x)$ such that $R_{ij;k} = \alpha_k R_{ij}$

$$\begin{aligned} 4 &= R_{11;1} = \alpha_1 R_{11} = -2\alpha_1, \quad & 0 &= R_{11;2} = \alpha_2 R_{11} = -2\alpha_2, \\ 4 &= R_{22;1} = \alpha_1 R_{22} = -\alpha_1, \quad & 4 &= R_{22;2} = \alpha_2 R_{22} = -\alpha_2 \quad \text{etc.} \end{aligned}$$

is inconsistent in our case. The connection is a non-metrizable one. There are no time- and velocities-independent multipliers g_{ij}.

Example 3 ([2]) The system

$$\ddot{x}^1 = 0, \quad \ddot{x}^2 = -2\dot{x}^1\dot{x}^2 \tag{28}$$

defines on $\mathbb{R}^2$ (or on $\mathbb{R} \times \mathbb{S}^1$, or on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$) a symmetric linear connection ∇ with Christoffel symbols $\Gamma^2_{12} = \Gamma^2_{21} = 1$, $\Gamma^k_{ij} = 0$ otherwise. We can easily check that Ric is symmetric, since $\psi_1 = \Gamma^1_{11} + \Gamma^2_{12} = 1$, and $\psi_2 = \Gamma^1_{21} + \Gamma^2_{22} = 0$. But it is degenerate, evaluation of the components brings $(R_{ij}) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore ∇ is not metrizable (even locally). If we try to solve directly the system corresponding to $\nabla g = 0$,

$$\begin{aligned} \partial_1 g_{11} &= 0, & \partial_1 g_{12} &= g_{12}, & \partial_1 g_{22} &= 2g_{22}, \\ \partial_2 g_{11} &= 2g_{12}, & \partial_2 g_{12} &= g_{22}, & \partial_2 g_{22} &= 0, \end{aligned}$$

we get a similar answer, $G = (g_{ij}) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Example 4 ([2]) Equations

$$\ddot{x}^1 = -(\dot{x}^1)^2, \qquad \ddot{x}^2 = -(\dot{x}^2)^2 \tag{29}$$

determine on $M_2 = \mathbb{R}^2$ a symmetric linear connection $\nabla_{X_1} X_1 = X_1 = 0$, $\nabla_{X_2} X_2 = X_2$, $\nabla_{X_i} X_j = 0$ otherwise, $X_i = \frac{\partial}{\partial x^i}$, with Christoffels

$$\Gamma^1_{11} = \Gamma^2_{22} = 1, \qquad \Gamma^k_{ij} = 0 \quad \text{otherwise.}$$

The curvature tensor R vanishes, equivalently, Ric $= 0$, the connection ∇ is flat, hence (locally) metrizable, and the system (29) is variational. To find out components of the metric, or another speaking, variational multipliers g_{ij}, we can solve the system of PDEs

$$\begin{aligned} &\partial_1 g_{11} = 2g_{11}, \quad &&\partial_1 g_{12} = g_{12}, \quad &&\partial_1 g_{22} = 0, \\ &\partial_2 g_{11} = 0, &&\partial_2 g_{12} = g_{12}, &&\partial_2 g_{22} = 2g_{22}. \end{aligned}$$

Given $x_0 \in M$, a non-singular 2×2 matrix (g^0_{ij}) and initial data $g_{ij}(x_0) = g^0_{ij}$ the solution is $g_{11} = g^0_{11} e^{2x^1}$, $g_{12} = g^0_{12} e^{x^1 + x^2}$, $g_{22} = g^0_{22} e^{2x^2}$, hence we get a (global) metric on $\mathbb{R}^2$ and the corresponding Lagrangian,

$$g_{ij} = g^0_{ij} \cdot e^{x^i + x^j}, \qquad L = \frac{1}{2} g^0_{ij} e^{x^i + x^j} \dot{x}^i \dot{x}^j$$

(remark that direct search for solution of the corresponding system of PDEs need not be easy in most cases). The Ricci tensor brings the same answer.

Note that if we introduce essentially the same connection on the “infinite cylindr” $\mathbb{S}^1 \times \mathbb{R}$, or on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, such a connection is not globally metrizable. Indeed, consider the (continuous, even smooth) function $f(t) = |X_1(\gamma(t))|$, $t \in (0,1)$, the length of the (smooth and globally defined) coordinate vector field X_1 along the “flow line” (which is the circle without one point): it satisfies $f' = 2f$; the metric behaves “exponentially”. We must expect problems with successful “taping” of the metric on the overlap of coordinate neighborhoods.

Another example of C^∞-connection which is metrizable locally but not globally is given in [16], cf. [22].

Example 5 For the system

$$\ddot{x}^1 + \dot{x}^1 \dot{x}^2 = 0, \qquad \ddot{x}^2 - \frac{1}{2} \exp(x^2)(\dot{x}^1)^2 = 0, \tag{30}$$

non-zero components are $\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}$, $\Gamma^2_{11} = -\frac{1}{2} e^{x^2}$. The Ricci tensor with components Ric $= -\frac{1}{4} e^{x^2} dx^1 \otimes dx^1 - \frac{1}{4} dx^2 \otimes dx^2$ is covariant constant, ∇Ric $= 0$, therefore recurrent with vanishing (and consequently exact) 1-form $\omega = 0 = d\,(\text{const})$ entirely on $\mathbb{R}^2$. All (global) compatible metrics on $\mathbb{R}^2$ form a one-parameter family

$$g_b = \exp{(x^2 + b)}\, dx^1 \otimes dx^1 + \exp{(b)}\, dx^2 \otimes dx^2, \quad b \in \mathbb{R}, \tag{31}$$

which yields Lagrangians $L = \frac{1}{2} e^{x^2 + b} (\dot{x}^1)^2 + \frac{1}{2} e^b (\dot{x}^2)^2$.

References

[1] Boothby, W. M.: An Introduction to Differentiable Manifolds and Riemannian Geometry. *Academic Press, Amsterdam–London–New York–Oxford–Paris–Tokyo,* 2003 (revised second editon).

[2] Cocos, M.: *A note on symmetric connections.* J. Geom. Phys. **56** (2006), 337–343.

[3] do Carmo, M. P.: Riemannian Geometry. *Birkhäuser, Boston–Basel–Berlin,* 1992.

[4] Cheng, K. S, Ni, W. T.: *Necessary and sufficient conditions for the existence of metrics in two-dimensional affine manifolds.* Chinese J. Phys. **16** (1978), 228–232.

[5] Douglas, J.: *Solution of the inverse problem of the calculus of variations.* Trans. AMS **50** (1941), 71–128.

[6] Dodson, C. T. J., Poston, T.: Tensor Geometry. The Geometric Viewpoint and its Uses. *Spriger, New York–Berlin–Heidelberg,* 1991 (second editon).

[7] Eisenhart, L. P., Veblen, O.: *The Riemann geometry and its generalization.* Proc. London Math. Soc. **8** (1922), 19–23.

[8] Jost, J.: Riemannian Geometry and Geometric Analysis. *Springer, Berlin–Heidelberg–New York,* 2005.

[9] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry I, II. *Wiley, New York–Chichester–Brisbane–Toronto–Singapore,* 1991.

[10] Kolář, I., Slovák, J., Michor, P. W.: Natural Operations in Differential Geometry. *Springer, Berlin–Heidelberg–New York,* 1993.

[11] Kowalski, O.: *On regular curvature structures.* Math. Z. **125** (1972), 129–138.

[12] Lovelock, D., Rund, H.: Tensors, Differential Forms, and Variational Principle. *Wiley, New York–London–Sydney,* 1975.

[13] Mikeš, J., Kiosak, V., Vanžurová, A.: Geodesic Mappings of Manifolds with Affine Connection. *Palacký Univ. Publ., Olomouc,* 2008.

[14] Nomizu, K., Sasaki, T.: Affine Differential Geometry. Geometry of Affine Immersions. *Cambridge Univ. Press, Cambridge,* 1994.

[15] Petrov, A. Z.: Einstein Spaces. *Moscow,* 1961 (in Russian).

[16] Schmidt, B. G.: *Conditions on a connection to be a metric connection.* Commun. Math. Phys. **29** (1973), 55–59.

[17] Sinyukov, N. S.: Geodesic Mappings of Riemannian Spaces. *Moscow,* 1979 (in Russian).

[18] Thompson, G.: *Local and global existence of metrics in two-dimensional affine manifolds.* Chinese J. Phys. **19**, 6 (1991), 529–532.

[19] Vanžurová, A.: *Linear connections on two-manifolds and SODEs.* Proc. Conf. Aplimat 2007 (Bratislava, Slov. Rep.), Part II (2007), 325–332.

[20] Vanžurová, A.: *Metrization problem for linear connections and holonomy algebras.* Archivum Mathematicum (Brno) **44** (2008), 339–348.

[21] Vanžurová, A.: *Metrization of linear connections, holonomy groups and holonomy algebras.* Acta Physica Debrecina **42** (2008), 39–48.

[22] Vanžurová, A., Žáčková, P.: *Metrization of linear connections.* Aplimat, J. of Applied Math. (Bratislava) **2**, 1 (2009), 151–163.

[23] Wolf, J. A.: Spaces of Constant Curvature. *Berkley, California,* 1972.