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Some Common Fixed Point Theorems in Normed Linear Spaces

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Abstract

In this paper, we establish some generalizations to approximate common fixed points for selfmappings in a normed linear space using the modified Ishikawa iteration process with errors in the sense of Liu [10] and Rafiq [14]. We use a more general contractive condition than those of Rafiq [14] to establish our results. Our results, therefore, not only improve a multitude of common fixed point results in literature but also generalize some of the results of Berinde [3], Rhoades [15] and recent results of Rafiq [14].

Key words: Common fixed point, contractive condition, Mann and Ishikawa iterations.

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1 Introduction

Let K be a nonempty closed convex subset of a normed linear space E and $T\colon K \to K$ a selfmap. For arbitrary x_0 in K, we define Mann [11] iteration process $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = (1-b_n)x_n + b_nTx_n, \quad n = 0, 1, 2, \dots \tag{1}$$

Ishikawa [6] iteration process $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_nTy_n \\ y_n &= (1-b_n^{'})x_n + b_n^{'}Tx_n, \quad n = 0, 1, 2, \dots \end{aligned} \tag{2}$$

where $x_0 \in K$ is arbitrary, $\{b_n\}$ and $\{b_n^{'}\}$ being sequences of real numbers in [0,1].

The concept of Ishikawa iteration process with errors was introduced by Liu [10] and is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_n T y_n + u_n \\ y_n &= (1-b_n^{'})x_n + b_n^{'} T x_n + v_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{3}$$

where $x_0 \in K$ is arbitrary, $\{b_n\}$ and $\{b_n^{'}\}$ being sequences of real numbers in [0,1] while $\{u_n\}$ and $\{v_n\}$ satisfy

$$\sum_{n=0}^{\infty} \|u_n\| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|v_n\| < \infty$$

respectively. We observe that (3) contains (1) and (2). We also observe that (3) contains the Mann iteration process with errors given by

$$x_{n+1} = (1-b_n)x_n + b_n T x_n + u_n, \quad n = 0,1,2,\ldots \tag{4}$$

Das and Debata [5] generalized the Ishikawa iteration processes from the case of one self mapping to the case of two self mappings S and T of K given by

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_n S y_n \\ y_n &= (1-b_n^{'})x_n + b_n^{'} T x_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{5}$$

By using Iteration (5), Das and Debata [5] established the common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space. Several other researchers such as Takahashi and Tamura [21] investigated iteration (5) in a strictly convex Banach space, for the case of two nonexpansive mappings under different assumptions and contractive conditions.

Later, Rafiq [14] studied the two-step iteration process with errors in the sense of Liu [10] by using the following sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= b_n S y_n + (1-b_n)x_n + u_n \\ y_n &= b_n^{'} T x_n + (1-b_n^{'})x_n + v_n, \quad n = 0,1,2,\ldots \end{aligned} \tag{6}$$

where $x_0 \in K$ is arbitrary, $\{u_n\}$ and $\{v_n\}$ are two summable sequences in K.

We observe that iteration (6) contains all the iteration processes (1)–(5) as special cases.

In 1972, Zamfirescu [23] proved the following result.

Theorem 1 *Let (E,d) be a complete metric space and $T\colon E \to E$ be a mapping for which there exist real numbers a, b and c satisfying $0 \le a < 1$, $0 \le b$, $c < 0.5$ such that, for each $x, y \in E$, at least one of the following is true:*

(Z_1) $d(Tx,Ty) \le ad(x,y)$;

(Z_2) $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$;

(Z_3) $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$.

Then, T is a Picard mapping.

An operator T satisfying the contractive conditions $(Z_1), (Z_2)$ and (Z_3) in Theorem 1 above is called a Zamfirescu operator.

Remark 1 The proof of this Theorem is contained in Berinde [2]. Indeed, if

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \tag{7}$$

in Theorem 1, we obtain

$$0 \le \delta < 1. \tag{8}$$

Then, for all $x, y \in E$, and by using Z_2, it was proved in Berinde [2] that

$$d(Tx, Ty) \le 2\delta d(x, Tx) + \delta d(x, y), \tag{9}$$

and by using Z_3, we obtain

$$d(Tx, Ty) \le 2\delta d(x, Ty) + \delta d(x, y), \tag{10}$$

where $0 \le \delta < 1$ is as defined by (7).

Remark 2 If $(E, \|.\|)$ is a normed linear space, then (9) becomes

$$\|Tx - Ty\| \le 2\delta \|x - Tx\| + \delta \|x - y\|, \tag{11}$$

for all $x, y \in E$ and where $0 \le \delta < 1$ is as defined by (7).

In 2008, Rafiq [14] proved a convergence theorem and some corollaries to approximate common fixed points of quasi-contractive operators on a normed space by using iteration (6) and under the assumption that the two self mappings S and T satisfy the conditions of a Zamfirescu operator.

Our aim in this paper is to establish some common fixed point theorems by using a more general contractive condition than those of Rafiq [14]. We shall use iteration (6) and employ the following contractive definition: Let K be a nonempty closed convex subset of a normed linear space E and $T\colon K \to K$ a selfmap of K. There exist a constant $L \ge 0$ such that $\forall x, y \in K$, we have

$$\|Tx - Ty\| \le e^{L\|x - Tx\|}(2\delta \|x - Tx\| + \delta \|x - y\|), \tag{12}$$

where $0 \le \delta < 1$ is as defined by (7) and e^x denotes the exponential function of $x \in K$.

Remark 3 The contractive condition (12) is more general than those of Rafiq [14] and others in the following sense:
If $L = 0$ in the contractive condition (12), then we obtain

$$\|Tx - Ty\| \le 2\delta \|x - Tx\| + \delta \|x - y\|$$

which is the Zamfirescu contraction condition used by Rafiq [14], where

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \quad 0 \le \delta < 1,$$

while constants a, b and c are as defined in Theorem 1 above.

The following lemma contained in Liu [10] will be required in the sequel.

Lemma 1 *Let $\{\rho_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$\rho_{n+1} \leq (1-s_n)\rho_n + s_n t_n + k_n,$$

for all $n \geq 1$. If

$$\sum_{n=0}^{\infty} s_n = \infty, \quad \lim_{n\to\infty} t_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} k_n < \infty$$

hold, then

$$\lim_{n\to\infty} \rho_n = 0.$$

2 Main results

Theorem 2 *Let K be a nonempty closed convex subset of a normed linear space E. Suppose that $S, T\colon K \to K$ are two selfmappings of K satisfying the contractive condition (12). Suppose also that $\{x_n\}_{n=0}^{\infty}$ is a sequence defined iteratively by (6).*

Let $F_S \bigcap F_T \neq \phi$, where F_S and F_T are the sets of fixed points of S and T respectively.

If in iteration (6) we have,

$$\sum_{n=0}^{\infty} b_n = \infty, \quad \sum_{n=0}^{\infty} \|u_n\| < \infty \quad \text{and} \quad \lim_{n\to\infty} \|v_n\| = 0,$$

then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of S and T.

Proof Since S and T satisfy the contractive definition (12), then for $x, y \in K$, we have

$$\|Sx - Sy\| \leq e^{L\|x-Sx\|}(2\delta\|x-Sx\| + \delta\|x-y\|) \tag{13}$$

and

$$\|Tx - Ty\| \leq e^{L\|x-Tx\|}(2\delta\|x-Tx\| + \delta\|x-y\|) \tag{14}$$

where $L \geq 0$ and $0 \leq \delta < 1$ is as defined by (7).

By assumption, $F_S \bigcap F_T \neq \phi$. Let $p \in F_S \bigcap F_T$.

Therefore, for arbitrary $x_0 \in K$ and by using iteration process (6), we get

$$\begin{aligned} x_{n+1} - p &= (1-b_n)x_n + b_n S y_n + u_n - p \\ &= (1-b_n)x_n + b_n S y_n - b_n p - (1-b_n)p + u_n \\ &= (1-b_n)(x_n - p) + b_n(Sy_n - p) + u_n \end{aligned}$$

and hence,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-b_n)(x_n-p) + b_n(Sy_n - p) + u_n\| \\ &\leq (1-b_n)\|x_n - p\| + b_n\|Sy_n - p\| + \|u_n\| \\ &= (1-b_n)\|x_n - p\| + b_n\|Sy_n - Sp\| + \|u_n\| \\ &= (1-b_n)\|x_n - p\| + b_n\|Sp - Sy_n\| + \|u_n\| \end{aligned}$$

By using (13), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad \leq (1 - b_n) \|x_n - p\| + b_n[e^{L\|p - Sp\|}(2\delta \|p - Sp\| + \delta \|p - y_n\|)] + \|u_n\| \\
&\quad = (1 - b_n) \|x_n - p\| + b_n[e^{L\|p - p\|}(2\delta \|p - p\| + \delta \|y_n - p\|)] + \|u_n\| \\
&\quad = (1 - b_n) \|x_n - p\| + b_n[e^{L(0)}(2\delta(0) + \delta \|y_n - p\|)] + \|u_n\| \\
&\quad = (1 - b_n) \|x_n - p\| + b_n[e^{0}(0 + \delta \|y_n - p\|)] + \|u_n\| \\
&\quad = (1 - b_n) \|x_n - p\| + b_n\delta \|y_n - p\| + \|u_n\|
\end{aligned}$$

Therefore,

$$\|x_{n+1} - p\| \leq (1 - b_n) \|x_n - p\| + b_n\delta \|y_n - p\| + \|u_n\| . \tag{15}$$

Similarly, by using iteration process (6), we obtain

$$\begin{aligned}
\|y_n - p\| &= \|(1 - b'_n)(x_n - p) + b'_n(Tx_n - p) + v_n\| \\
&\leq (1 - b'_n) \|x_n - p\| + b'_n \|Tx_n - p\| + \|v_n\| \\
&= (1 - b'_n) \|x_n - p\| + b'_n \|Tx_n - Tp\| + \|v_n\| \\
&= (1 - b'_n) \|x_n - p\| + b'_n \|Tp - Tx_n\| + \|v_n\|
\end{aligned}$$

By using (14), we get

$$\begin{aligned}
&\|y_n - p\| \\
&\quad \leq (1 - b'_n) \|x_n - p\| + b'_n[e^{L\|p - Tp\|}(2\delta \|p - Tp\| + \delta \|p - x_n\|)] + \|v_n\| \\
&\quad = (1 - b'_n) \|x_n - p\| + b'_n[e^{L\|p - p\|}(2\delta \|p - p\| + \delta \|x_n - p\|)] + \|v_n\| \\
&\quad = (1 - b'_n) \|x_n - p\| + b'_n[e^{L(0)}(2\delta(0) + \delta \|x_n - p\|)] + \|v_n\| \\
&\quad = (1 - b'_n) \|x_n - p\| + b'_n[e^{0}(0 + \delta \|x_n - p\|)] + \|v_n\| \\
&\quad = (1 - b'_n) \|x_n - p\| + b'_n\delta \|x_n - p\| + \|v_n\|
\end{aligned}$$

which implies that

$$\|y_n - p\| \leq (1 - b'_n + b'_n\delta) \|x_n - p\| + \|v_n\| . \tag{16}$$

By observing that $0 \leq b'_n \leq 1,\ 0 \leq \delta < 1$ and since $0 \leq (1 - b'_n + b'_n\delta) < 1$, we obtain

$$\|y_n - p\| \leq \|x_n - p\| + \|v_n\| . \tag{17}$$

Substitute (17) into (15) yields

$$\|x_{n+1} - p\| \leq (1 - b_n) \|x_n - p\| + b_n\delta \|x_n - p\| + b_n\delta \|v_n\| + \|u_n\| .$$

and hence,

$$\|x_{n+1} - p\| \leq (1 - b_n + b_n\delta) \|x_n - p\| + b_n\delta \|v_n\| + \|u_n\| . \tag{18}$$

By applying Lemma 1 and using the fact that

$$0 \leq b_n \leq 1, \quad 0 \leq \delta < 1, \quad 0 \leq (1 - b_n + b_n\delta) < 1,$$
$$\sum_{n=0}^{\infty} b_n = \infty, \quad \sum_{n=0}^{\infty} \|u_n\| < \infty \quad \text{and} \quad \lim_{n\to\infty} \|v_n\| = 0,$$

we obtain

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0$$

which implies that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of S and T.

This completes the proof. □

Remark 4 Our result in Theorem 2 is a generalization of Theorem 2.1 of Rafiq [14].

Theorem 3 *Let K be a nonempty closed convex subset of a normed linear space E. Suppose that $S\colon K \to K$ is a selfmap of K satisfying the contractive condition (12). Suppose also that $\{x_n\}_{n=0}^{\infty}$ is a sequence defined iteratively by (4).*

Let F_S be the set of fixed points of S such that $F_S \neq \phi$. If in iteration (4) we have, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty} \|u_n\| < \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of S.

Proof By assumption, $F_S \neq \phi$. Let $p \in F_S$. Therefore, for arbitrary $x_0 \in K$ and by using iteration process (4), we get

$$\begin{aligned} x_{n+1} - p &= (1 - b_n)x_n + b_n Sx_n + u_n - p \\ &= (1 - b_n)x_n + b_n Sx_n - b_n p - (1 - b_n)p + u_n \\ &= (1 - b_n)(x_n - p) + b_n(Sx_n - p) + u_n \end{aligned}$$

and hence,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - b_n)(x_n - p) + b_n(Sx_n - p) + u_n\| \\ &\leq (1 - b_n)\|x_n - p\| + b_n\|Sx_n - p\| + \|u_n\| \\ &= (1 - b_n)\|x_n - p\| + b_n\|Sx_n - Sp\| + \|u_n\| \\ &= (1 - b_n)\|x_n - p\| + b_n\|Sp - Sx_n\| + \|u_n\| \end{aligned}$$

Since S satisfies the contractive condition (12), we get

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\leq (1 - b_n)\|x_n - p\| + b_n[e^{L\|p-Sp\|}(2\delta\|p - Sp\| + \delta\|p - x_n\|)] + \|u_n\| \\ &= (1 - b_n)\|x_n - p\| + b_n[e^{L\|p-p\|}(2\delta\|p - p\| + \delta\|x_n - p\|)] + \|u_n\| \\ &= (1 - b_n)\|x_n - p\| + b_n[e^{L(0)}(2\delta(0) + \delta\|x_n - p\|)] + \|u_n\| \\ &= (1 - b_n)\|x_n - p\| + b_n[e^{0}(0 + \delta\|x_n - p\|)] + \|u_n\| \\ &= (1 - b_n)\|x_n - p\| + b_n\delta\|x_n - p\| + \|u_n\| \end{aligned}$$

and hence,

$$\|x_{n+1} - p\| \leq (1 - b_n + b_n\delta)\|x_n - p\| + \|u_n\|.$$

By using Lemma 1 and the fact that

$$0 \leq b_n \leq 1, \quad 0 \leq \delta < 1, \quad 0 \leq (1 - b_n + b_n\delta) < 1,$$
$$\sum_{n=0}^{\infty} b_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|u_n\| < \infty,$$

we obtain

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0$$

which implies that $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of S.

To prove the uniqueness, we take $p_1, p_2 \in F_S$ and assume that $p_1 \neq p_2$.

By using the contractive condition (12) and $0 \leq \delta < 1$, we get

$$\begin{aligned}
\|p_1 - p_2\| &= |Sp_1 - Sp_2\| \\
&\leq e^{L\|p_1 - Sp_1\|}(2\delta\|p_1 - Sp_1\| + \delta\|p_1 - p_2\|) \\
&= e^{L\|p_1 - p_1\|}(2\delta\|p_1 - p_1\| + \delta\|p_1 - p_2\|) \\
&= e^{L(0)}(2\delta(0) + \delta\|p_1 - p_2\|) \\
&= e^{0}(0 + \delta\|p_1 - p_2\|) \\
&= \delta\|p_1 - p_2\|) \\
&< \|p_1 - p_2\|
\end{aligned}$$

which is a contradiction. Hence, $p_1 = p_2$.

This completes the proof. □

Remark 5 The uniqueness result in Theorem 3 is a generalization of Corollary 2.2 of Rafiq [14].

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