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Quotients and Homomorphisms of Relational Systems*

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Abstract

Relational systems containing one binary relation are investigated. Quotient relational systems are introduced and some of their properties are characterized. Moreover, homomorphisms, strong mappings and cone preserving mappings are introduced and the interplay between these notions is considered. Finally, the connection between directed relational systems and corresponding groupoids is investigated.

Key words: Relational system, quotient relational system, cone, homomorphism, strong mapping, cone preserving mapping, groupoid, g-homomorphism, quotient groupoid.

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1 Introduction

The theory of relational systems and of binary relations was settled in the pioneering work by J. Riguet ([6]) and the classical algebraic approach by A. I. Mal'cev ([5]). Several properties of equivalences and homomorphisms were treated by the first author in [2] for equivalence systems, in [4] for quasiordered

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sets and in [3] for posets. It turns out that these results can be extended to arbitrary relational systems. This is the aim of the present paper. In particular, we set up several properties of quotient relational systems and relate the concept of strong homomorphism to that of cone preserving mapping.

We start with the definition of a relational system.

Definition 1.1 By a *relational system* we mean an ordered pair $\mathcal{A} = (A, R)$ consisting of a set A and a binary relation R on A. $\mathcal{A}$ is called *connex* if for all $a, b \in A$ either $(a, b) \in R$ or $(b, a) \in R$ (or both). $\mathcal{A}$ is called *directed* if $U_R(a, b) \neq \emptyset$ for all $a, b \in A$ where

$$U_R(a, b) := \{c \in A \mid (a, c), (b, c) \in R\}$$

is called the *(upper) cone* of $\{a, b\}$ with respect to R. If $a = b$ then $U_R(a, b)$ is simply denoted by $U_R(a)$.

2 Quotient relational systems

In this section we investigate quotients of relational systems by equivalence relations. The aim is to show what properties of a given relation are preserved under factorization by means of an equivalence.

First we define a quotient relational system as follows:

Definition 2.1 Let $\mathcal{A} = (A, R)$ be a relational system and Θ an equivalence relation on A. Then $\mathcal{A}/\Theta := (A/\Theta, R/\Theta)$ where

$$R/\Theta := \{([a]\Theta, [b]\Theta) \bigm| (a, b) \in R\}$$

is called the *quotient relational system* of $\mathcal{A}$ by Θ.

The following lemma will be useful in the proof of the next theorem.

Lemma 2.2 *Let $\mathcal{A} = (A, R)$ be a relational system, $a, b \in A$ and Θ an equivalence relation on A. Then the following are equivalent:*

(i) $([a]\Theta, [b]\Theta) \in R/\Theta$

(ii) $(a, b) \in \Theta \circ R \circ \Theta$

Proof Both (i) and (ii) are equivalent to the fact that there exists $(c, d) \in R$ with $([c]\Theta, [d]\Theta) = ([a]\Theta, [b]\Theta)$. □

Now we characterize some properties of the relation R/Θ.

Theorem 2.3 *Let $\mathcal{A} = (A, R)$ be a relational system and Θ an equivalence relation on A. Then (i)–(v) hold:*

(i) *R/Θ is reflexive if and only if $([a]\Theta)^2 \cap R \neq \emptyset$ for all $a \in A$.*

(ii) *R/Θ is symmetric if and only if $R \subseteq \Theta \circ R^{-1} \circ \Theta$.*

(iii) *R/Θ is antisymmetric if and only if $(\Theta \circ R \circ \Theta) \cap (\Theta \circ R^{-1} \circ \Theta) \subseteq \Theta$.*

(iv) *R/Θ is transitive if and only if $R \circ \Theta \circ R \subseteq \Theta \circ R \circ \Theta$.*

(v) *R/Θ is connex if and only if so is $\Theta \circ R \circ \Theta$.*

Proof Let $a, b \in A$. We use Lemma 2.2.

(i) R/Θ is reflexive if and only if $(a, a) \in \Theta \circ R \circ \Theta$ for all $a \in A$.

(ii) R/Θ is symmetric if and only if $\Theta \circ R \circ \Theta$ has this property.

(iii) R/Θ is antisymmetric if and only if $(a, b) \in \Theta \circ R \circ \Theta$ and $(b, a) \in \Theta \circ R \circ \Theta$ together imply $(a, b) \in \Theta$.

(iv) The following are equivalent:
R/Θ is transitive.
$\Theta \circ R \circ \Theta$ is transitive.
$\Theta \circ R \circ \Theta \circ \Theta \circ R \circ \Theta \subseteq \Theta \circ R \circ \Theta$.
$\Theta \circ R \circ \Theta \circ R \circ \Theta \subseteq \Theta \circ R \circ \Theta$.
$R \circ \Theta \circ R \subseteq \Theta \circ R \circ \Theta$.

(v) This is clear. □

Corollary 2.4 *Let $\mathcal{A} = (A, R)$ be a relational system and Θ an equivalence relation on A. Then (i)–(iii) hold:*

(i) *If R is reflexive then so is R/Θ.*

(ii) *If R is symmetric then so is R/Θ.*

(iii) *If R is transitive and either $\Theta \circ R \subseteq R \circ \Theta$ or $R \circ \Theta \subseteq \Theta \circ R$ (or both) then R/Θ is transitive, too.*

Remark 2.5 It was proved in Theorem 2 of [1] that if R is transitive then R/Θ need not have this property.

3 Homomorphisms of relational systems

In this section we investigate the interplay between different properties of mappings between relational systems. The aim is to characterize strong and cone preserving mappings and homomorphisms by means of the relational products with the induced equivalence.

In the following, if $\mathcal{A} = (A, F)$ and $\mathcal{B} = (B, S)$ are relational systems and $f: A \to B$ a map from A to B then

$$f^{-1}(S) := \{(a, b) \in A^2 \mid (f(a), f(b)) \in S\}.$$

Moreover, if Θ is an equivalence relation on A then f_Θ denotes the canonical mapping $x \mapsto [x]\Theta$ from A to A/Θ.

Definition 3.1 (cf. [4] and [5]) Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and $f\colon A \to B$.

(i) f is called a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $(f(a), f(b)) \in S$ for all $(a, b) \in R$.

(ii) f is called *strong* if for all $(c, d) \in S$ there exists $(a, b) \in R$ with $f(a) = c$ and $f(b) = d$.

(iii) f is called *cone preserving* if $f(U_R(a)) = U_S(f(a))$ for all $a \in A$.

Remark 3.2 (iii) extends the concept defined in [4] for quasiordered sets. Obviously, f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if and only if $f(U_R(a)) \subseteq U_S(f(a))$ for all $a \in A$. Hence, if f is cone preserving then it is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Lemma 3.3 *Let $\mathcal{A} = (A, F)$ be a relational system and Θ an equivalence relation on A. Then f_Θ is a strong homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\Theta$ and R/Θ is the smallest binary relation T on A/Θ such that f_Θ is a homomorphism from $\mathcal{A}$ to $(A/\Theta, T)$.*

Proof The proof is straightforward. □

For all sets A, B and every map $f\colon A \to B$ let $\ker f$ denote the equivalence relation on A induced by f.

Now we state some characterizations of the notion of homomorphism which will be used later.

Lemma 3.4 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and $f\colon A \to B$. Then the following are equivalent:*

(i) *f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

(ii) $f^{-1}(S) \supseteq R$

(iii) $f^{-1}(S) \supseteq (\ker f) \circ R$

(iv) $f^{-1}(S) \supseteq R \circ (\ker f)$

(v) $f^{-1}(S) \supseteq (\ker f) \circ R \circ (\ker f)$

Proof (i) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (i) □

Lemma 3.5 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a mapping from A onto B. Then the following are equivalent:*

(i) *f is strong.*

(ii) $f^{-1}(S) \subseteq (\ker f) \circ R \circ (\ker f)$

Proof Both are equivalent to the fact that for all $a, b \in A$ with $(f(a), f(b)) \in S$ there exists a $(c, d) \in R$ with $f(c) = f(a)$ and $f(d) = f(b)$. $\square$

Theorem 3.6 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then the following are equivalent:*

(i) *f is strong.*

(ii) $f^{-1}(S) = (\ker f) \circ R \circ (\ker f)$

Proof This follows from Lemmata 3.4 and 3.5. $\square$

Lemma 3.7 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a mapping from A onto B. Then the following are equivalent:*

(i) *f is cone preserving.*

(ii) $f^{-1}(S) = R \circ (\ker f)$

Proof The following are equivalent:

$f(U_R(a)) = U_S(f(a))$ for all $a \in A$

For all $a, b \in A$, $f(b) \in U_S(f(a))$ if and only if $f(b) \in f(U_R(a))$.

For all $a, b \in A$, $f(b) \in U_S(f(a))$ if and only if there exists a $c \in U_R(a)$ with $f(c) = f(b)$. $\square$

Theorem 3.8 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then the following are equivalent:*

(i) *f is cone preserving.*

(ii) $f^{-1}(S) \subseteq R \circ (\ker f)$

Proof This follows from Lemmata 3.4 and 3.7. $\square$

Theorem 3.9 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a strong homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then the following are equivalent:*

(i) *f is cone preserving.*

(ii) $(\ker f) \circ R \subseteq R \circ (\ker f)$

Proof (i) $\Rightarrow$ (ii): This follows from Lemmata 3.7 and 3.4.

(ii) $\Rightarrow$ (i):

$$f^{-1}(S) \subseteq (\ker f) \circ R \circ (\ker f) \subseteq R \circ (\ker f) \circ (\ker f) = R \circ (\ker f)$$

and hence f is cone preserving according to Theorem 3.8. $\square$

Corollary 3.10 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems, assume R to be symmetric and let f be a strong homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then the following are equivalent:*

(i) *f is cone preserving.*

(ii) $(\ker f) \circ R = R \circ (\ker f)$

Proof (i) $\Rightarrow$ (ii): According to Theorem 3.9, $(\ker f) \circ R \subseteq R \circ (\ker f)$. Since R is symmetric, we have

$$\begin{aligned} R \circ (\ker f) &= R^{-1} \circ (\ker f)^{-1} = ((\ker f) \circ R)^{-1} \subseteq (R \circ (\ker f))^{-1} \\ &= (\ker f)^{-1} \circ R^{-1} = (\ker f) \circ R \end{aligned}$$

(ii) $\Rightarrow$ (i): This follows from Theorem 3.9. □

Corollary 3.11 *Let $\mathcal{A} = (A, R)$ be a relational system and Θ an equivalence relation on A. Then the following are equivalent:*

(i) *f_Θ is cone preserving.*

(ii) $\Theta \circ R \subseteq R \circ \Theta$

Proof This follows from Lemma 3.3 and Theorem 3.9. □

Corollary 3.12 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a cone preserving mapping from A onto B. Then f is strong.*

Proof This follows from Lemmata 3.7 and 3.5. □

4 Groupoids corresponding to relational systems

In this section we associate to certain relational systems certain groupoids and investigate relationships between homomorhisms and congruence relations of certain relational systems on the one hand and of the corresponding groupoids on the other hand.

Definition 4.1 Let $\mathcal{A} = (A, R)$ be a relational system. We say that a *groupoid* $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ *corresponds* to $\mathcal{A}$ if $ab = b$ in case $(a, b) \in R$ and $ab \in U_R(a, b)$ otherwise.

Remark 4.2 Every relational system $\mathcal{A} = (A, R)$ can be extended to a directed one by adjoining one element: If $1 \notin A$ then put $A_d := A \cup \{1\}$ and $R_d := R \cup (A_d \times \{1\})$. Then $\mathcal{A}_d := (A_d, R_d)$ is directed and $\mathcal{A}$ is the restriction of $\mathcal{A}_d$ to A.

Remark 4.3 Every relational system $\mathcal{A} = (A, R)$ can be considered as a directed graph with vertex-set A and edge-set R. For $1 \notin A$ the groupoid $(A, \cdot)$ defined by $ab := b$ if $(a, b) \in R$ and $ab := 1$ otherwise is called the corresponding *graph algebra*. It is in fact a groupoid corresponding to $\mathcal{A}_d$. Hence the groupoids corresponding to relational systems are generalizations of graph algebras.

Remark 4.4 Although a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to a relational system $\mathcal{A} = (A, R)$ need not be uniquely determined by $\mathcal{A}$, for $a, b \in A$, $ab = b$ if and only if $(a, b) \in R$.

In what follows, if we consider a groupoid $\mathcal{G}$ corresponding to a relational system $\mathcal{A} = (A, R)$ we assume that $\mathcal{G}$ exists, it means that $\mathcal{A}$ is (partly) directed, i.e. if $a, b \in A$ and $(a, b) \notin R$ then the upper cone $U_R(a, b)$ of $\{a, b\}$ with respect to R is not empty.

Theorem 4.5 *Let $\mathcal{A} = (A, R)$ be a relational system and $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ a groupoid corresponding to $\mathcal{A}$. Then (i)–(vi) hold:*

(i) *R is reflexive if and only if $\mathcal{G}$ is idempotent.*

(ii) *R is symmetric if and only if $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = x$.*

(iii) *If $\mathcal{G}(\mathcal{A})$ is commutative then R is antisymmetric.*

(iv) *If $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = xy$ then R is antisymmetric.*

(v) *If $\mathcal{G}(\mathcal{A})$ is a semigroup then R is transitive.*

(vi) *R is transitive if and only if $\mathcal{G}(\mathcal{A})$ satisfies the identity $x((xy)z) = (xy)z$.*

Proof (i), (iii) and (v) follow immediately from Remark 4.4.

Let $a, b, c \in A$.

(ii): First assume R to be symmetric. We have $(a, ab) \in R$ and hence, by symmetry of R, $(ab, a) \in R$ whence $(ab)a = a$. If, conversely, $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = x$ and if $(a, b) \in R$ then $ab = b$ and hence $ba = (ab)a = a$ which shows $(b, a) \in R$ proving symmetry of R.

(iv): If $(a, b), (b, a) \in R$ then $ab = b$ and $ba = a$ and hence $a = ba = (ab)a = ab = b$.

(vi): First assume R to be transitive. Since $(a, ab), (ab, (ab)c) \in R$ one has, by transitivity of R, $(a, (ab)c) \in R$ whence $a((ab)c) = (ab)c$. Conversely, assume $\mathcal{G}(\mathcal{A})$ to satisfy the identity $x((xy)z) = (xy)z$ and assume $(a, b), (b, c) \in R$. Then $ab = b$ and $bc = c$ and hence $ac = a(bc) = a((ab)c) = (ab)c = bc = c$ whence $(a, c) \in R$. □

Now we define the notion of a g-homomorphism.

Definition 4.6 Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Then f is called a *g-homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that $a, b, c, d \in A$, $f(a) = f(c)$ and $f(b) = f(d)$ together imply $f(ab) = f(cd)$.

The following theorem states that homomorphisms between groupoids corresponding to certain relational systems are g-homomorphisms between these relational systems.

Theorem 4.7 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems, $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ respectively $\mathcal{G}(\mathcal{B}) = (B, \circ)$ corresponding groupoids and f a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$. Then f is a g-homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

Proof Let $a, b, c, d \in A$. If $(a, b) \in R$ then $ab = b$ and hence $f(a) \circ f(b) = f(ab) = f(b)$ showing $(f(a), f(b)) \in S$. Moreover, if $a, b, c, d \in A$, $f(a) = f(c)$ and $f(b) = f(d)$ then $f(ab) = f(a) \circ f(b) = f(c) \circ f(d) = f(cd)$. □

In the next theorem we show that in some sense homomorphisms between relational systems are homomorphisms between corresponding groupoids.

Theorem 4.8 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems and f a strong g-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ with the groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$. Then there exists a groupoid $\mathcal{G}(\mathcal{B}) = (B, \circ)$ corresponding to $\mathcal{B}$ such that f is a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$.*

Proof According to Definition 4.6 there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that for each $a, b, c, d \in A$, if $f(a) = f(c)$ and $f(b) = f(d)$ then $f(ab) = f(cd)$. Define $f(x) \circ f(y) := f(xy)$ for all $x, y \in A$. According to Definition 4.6, $\circ$ is well-defined. Let $a, b \in A$. If $(f(a), f(b)) \in S$ then, since f is strong, there exists $(c, d) \in R$ with $f(c) = f(a)$ and $f(d) = f(b)$. Now $f(a) \circ f(b) = f(ab) = f(cd) = f(d) = f(b)$ according to Definition 4.6. If $(f(a), f(b)) \notin S$ then $(a, b) \notin R$ according to Definition 4.6 and hence $ab \in U_R(a, b)$, i.e. $(a, ab), (b, ab) \in R$ whence $(f(a), f(a) \circ f(b)) = (f(a), f(ab)) \in S$ and $(f(b), f(a) \circ f(b)) = (f(b), f(ab)) \in S$, i.e. $f(a) \circ f(b) \in U_S(f(a), f(b))$. This shows that $\mathcal{G}(\mathcal{B})$ corresponds to $\mathcal{B}$. Finally, f is a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$ since $f(xy) = f(x) \circ f(y)$ for all $x, y \in A$. □

Our next theorem contains some assertions concerning factor groupoids.

Theorem 4.9 *Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems. Then (i) and (ii) hold:*

(i) *If f is a g-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that* $\ker f \in \operatorname{Con} \mathcal{G}(\mathcal{A})$.

(ii) *If $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ is a groupoid corresponding to $\mathcal{A}$ and $\Theta \in \operatorname{Con} \mathcal{G}(\mathcal{A})$ then f_Θ is a g-homomorphism from $\mathcal{A}$ to $\mathcal{A}/\Theta$.*

Proof Let $a, b, c, d \in A$.

(i) Obviously, $\ker f \in \operatorname{Equ} A$. According to Definition 4.6 there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that for each $a, b, c, d \in A$, if $f(a) = f(c)$ and $f(b) = f(d)$ then $f(ab) = f(cd)$, i. e. $(a, c), (b, d) \in \ker f$ implies $(ab, cd) \in \ker f$.

(ii) If $(a,b) \in R$ then $(f_\Theta(a), f_\Theta(b)) = ([a]\Theta, [b]\Theta) \in R/\Theta$. Moreover, if $f_\Theta(a) = f_\Theta(c)$ and $f_\Theta(b) = f_\Theta(d)$ then $[a]\Theta = f_\Theta(a) = f_\Theta(c) = [c]\Theta$ and $[b]\Theta = f_\Theta(b) = f_\Theta(d) = [d]\Theta$ and hence

$$f_\Theta(ab) = [ab]\Theta = [a]\Theta \cdot [b]\Theta = [c]\Theta \cdot [d]\Theta = [cd]\Theta = f_\Theta(cd).$$

□

Finally, we assign to each groupoid two relational systems.

Definition 4.10 For every groupoid $\mathcal{G} = (G, \cdot)$ we define two *corresponding relational systems* $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}^*(\mathcal{G})$ by

$$\mathcal{A}(\mathcal{G}) := (G, R(\mathcal{G})) \text{ respectively } \mathcal{A}^*(\mathcal{G}) := (G, R^*(\mathcal{G}))$$

where

$$R(\mathcal{G}) := \{(x,y) \in G^2 \mid xy = y\}$$

respectively

$$R^*(\mathcal{G}) := \bigcup_{x,y \in G} \{(x, xy), (y, xy)\}.$$

Lemma 4.11 *Let $\mathcal{G} = (G, \cdot)$ be a groupoid. Then $\mathcal{A}^*(\mathcal{G})$ is directed.*

Proof $U_{R^*(\mathcal{G})}(x,y) \supseteq \{xy\} \neq \emptyset$ for every $x, y \in G$. □

Lemma 4.12 *Let $\mathcal{A} = (A, R)$ be a relational system and $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ a corresponding groupoid. Then $\mathcal{A}(\mathcal{G}(\mathcal{A})) = \mathcal{A}$ and $R \subseteq R^*(\mathcal{G}(\mathcal{A}))$.*

Proof Let $a, b \in A$. If $(a,b) \in R(\mathcal{G}(\mathcal{A}))$ then $ab = b$. Since $\mathcal{G}(\mathcal{A})$ corresponds to $\mathcal{A}$ we have $(a,b) \in R$. If, conversely, $(a,b) \in R$ then $ab = b$ in $\mathcal{G}(\mathcal{A})$ and hence $(a,b) \in R(\mathcal{G}(\mathcal{A}))$ and $(a,b) = (a, ab) \in R^*(\mathcal{G}(\mathcal{A}))$. □

Remark 4.13 As mentioned above, a groupoid $\mathcal{G}(\mathcal{A})$ corresponding to the relational system $\mathcal{A} = (A, R)$ need not be uniquely determined by $\mathcal{A}$. However, Lemma 4.12 shows this not to be essential since $\mathcal{A}$ can be reconstructed from every groupoid $\mathcal{G}(\mathcal{A})$ corresponding to $\mathcal{A}$ due to the fact $\mathcal{A}(\mathcal{G}(\mathcal{A})) = \mathcal{A}$. Hence $\mathcal{G}(\mathcal{A})$ contains all the information on $\mathcal{A}$. Therefore one can switch between $\mathcal{A}$ and $\mathcal{G}(\mathcal{A})$ whenever it is suitable. Of course, it is more convenient to work with groupoids instead of relational systems since the theory of groupoids is more advanced.

Theorem 4.14 *Let $\mathcal{G} = (G, \cdot)$ and $\mathcal{H} = (H, \circ)$ be groupoids and f a homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Then f is a homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$ and from $\mathcal{A}^*(\mathcal{G})$ to $\mathcal{A}^*(\mathcal{H})$.*

Proof Let $a, b \in G$. If $(a,b) \in R(\mathcal{G})$ then $ab = b$ and hence $f(a) \circ f(b) = f(ab) = f(b)$, i.e. $(f(a), f(b)) \in R(\mathcal{H})$. This shows that f is a homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$. If, on the other hand, $(a,b) \in R^*(\mathcal{G})$ then there exist $e,h \in G$ with $(a,b) \in \{(e,eh),(h,eh)\}$. Now $f(e), f(h) \in H$ and

$$\begin{aligned}(f(a),f(b)) \in \{(f(e),f(eh)),(f(h),f(eh))\}\\ = \{(f(e),f(e)\circ f(h)),(f(h),f(e)\circ f(h))\}\end{aligned}$$

and hence $(f(a),f(b)) \in R^*(\mathcal{H})$ showing that f is a homomorphism from $\mathcal{A}^*(\mathcal{G})$ to $\mathcal{A}^*(\mathcal{H})$. □

Lemma 4.15 *Let $\mathcal{G} = (G,\cdot)$ be a groupoid satisfying the identities $x(xy) = y(xy) = xy$. Then $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$.*

Proof Let $a,b \in G$. If $(a,b) \in R(\mathcal{G})$ then $ab = b$. If $(a,b) \notin R(\mathcal{G})$ then $a(ab) = b(ab) = ab$ and hence $(a,ab),(b,ab) \in R(\mathcal{G})$. □

Theorem 4.16 *Let $\mathcal{G} = (G,\cdot)$ be a groupoid. Then the following are equivalent:*

(i) *There exists a relational system $\mathcal{A} = (G,R)$ with reflexive R such that $\mathcal{G}$ corresponds to $\mathcal{A}$.*

(ii) *$\mathcal{G}$ is idempotent and satisfies the identities $x(xy) = y(xy) = xy$.*

Proof Let $a,b \in G$.

(i) ⇒ (ii): Since $(b,b) \in R$, $bb = b$. If $(a,b) \in R$ then $ab = b$ and hence $a(ab) = ab$ and $b(ab) = bb = b = ab$. If $(a,b) \notin R$ then $(a,ab),(b,ab) \in R$ and hence $a(ab) = b(ab) = ab$.

(ii) ⇒ (i): Put $R := \{(x,y) \in G^2 \mid xy = y\} \cup \{(x,x) \mid x \in G\}$. Then R is reflexive. Moreover, if $(a,b) \in R$ then $ab = b$ or $a = b$. In the second case $ab = bb = b$. If $(a,b) \notin R$ then $a(ab) = b(ab) = ab$ and hence $(a,ab),(b,ab) \in R$. □

Remark 4.17 Let $\mathcal{G} = (G,\cdot)$ and $\mathcal{H} = (H,\circ)$ be groupoids and f a homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Then f need not be a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$ as can be seen from the following example.

Example 4.18 Put $\mathcal{G} := (\{-1,0,1\},\cdot)$ and $\mathcal{H} := (\{0,1\},\cdot)$ where $\cdot$ denotes the multiplication of integers and let f denote the mapping $x \mapsto |x|$ from $\{-1,0,1\}$ to $\{0,1\}$. Then f is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ and $R(\mathcal{G}) = \{(-1,0),(0,0),(1,-1),(1,0),(1,1)\}$. Let $(\{-1,0,1\},*)$ be a groupoid corresponding to $\mathcal{A}(\mathcal{G})$. Then for $x,y \in \{-1,0,1\}$, $1 * x = x$ and $x * y = 0$ otherwise. Now $f(-1) = f(1)$ but $f((-1)*(-1)) = f(0) = 0 \neq 1 = f(1) = f(1*1)$ and hence f is not a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$. Observe e.g. $0 \cdot (1 \cdot 1) = 0 \neq 1 = 1 \cdot 1$.

Theorem 4.19 *Let $\mathcal{G} = (G,\cdot)$ and $\mathcal{H} = (H,\circ)$ be groupoids, assume $\mathcal{G}$ to satisfy the identities $x(xy) = y(xy) = xy$ and let f be a homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Then f is a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$.*

Proof According to Lemma 4.15, $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$. If $a, b, c, d \in G$, $f(a) = f(c)$ and $f(b) = f(d)$ then $f(ab) = f(a) \circ f(b) = f(c) \circ f(d) = f(cd)$ and hence f is a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$. The rest follows from Theorem 4.14. □

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