Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **50**, 2 (2011) 89–94

Information Measure for Vague Symbols*

Milan MAREŠ

ÚTIA AV ČR, P.O. Box 18, 182 08 Praha 8 and
ÚAI-PŘF, JČU, Branišovská 31a, 370 05 České Budějovice
e-mail: mares@utia.cas.cz

Dedicated to Lubomír Kubáček on the occasion of his 80th birthday

(Received March 31, 2011)

Abstract

The structures of the fuzzy information theory are focused on the concept of fuzzy entropy, where the individual information of symbols is considered only implicitely. This paper aims to fill this gap and to study the concepts of fuzzy information. Special attention is paid to the typical fuzzy set theoretical paradigma of monotonicity of operations.

Key words: information source, alphabet, fuzzy information, vague information, information measures, symbol, fuzzy entropy

2010 Mathematics Subject Classification: 94A17, 62B86, 03E72

1 Introduction

The classical information theory, where the uncertainty is represented by randomness, was established by Shannon and Weaver in [8].

When the fuzzy set theory was presented in [9], the demand for a specific information theory dealing with vagueness has appeared (see, e.g. [1, 2, 4] and many others). Meanwhile Shannon built his theory on the concept of information connected with individual symbols of the source alphabet and then he defined the entropy of the entire source as the probabilistic mean value of the particular information values, the fuzzy information measuring is derived from the concept of the entire fuzzy source entropy.

In fact, the definitions of fuzzy entropy presented, e.g., in [1, 4] and in other works, include some implicite components closely analogous to probabilistic

*The research summarized in this paper was partly supported by the Grant Agency of Czech Republic, project No. P402/11/0378, and by the Ministry of Education, Youth and Sports of the CR, Research Centre No. 1M0572.

information measure. But their specific character is not explicitely stressed and their properties are not deeply analyzed.

Moreover, the definitions of fuzzy entropy formulated in the referred works very closely repeat their classical probabilistic counterpart. As mentioned in [6, 7], this approach to the fuzzy information is not unavoidable, and the concept of fuzzy information can be significantly simplified.

Such simplified fuzzy information concept was suggested in [7], and its structure is analyzed and characterized in the following sections of this paper.

2 Information sources

In the whole paper, we denote by A the non-empty and finite *alphabet.* Its elements $a, a', a_1, a_2, \dots$ are called *symbols*, and we call any finite sequence of symbols by the term *word.* By $A^* = A \cup A^2 \cup A^3 \cup \dots \cup A^n \cup \dots$ we denote the set of all words from the alphabet A.

2.1 Probabilistic information measures

Due to the Shannon and Weaver model [8], the *random information source* is defined as a pair (A, p), where A is an alphabet and p is a probability distribution over A.

The information transmitted by a symbol $a \in A$ is defined as a number

$$I_p(a) = \log_2(1/p(a)) = -\log_2 p(a). \tag{1}$$

The uncertainty of the entire source (A, p) is measured by its entropy $H(A, p)$ defined as the mean value of symbol information

$$H(A, p) = \sum_{a \in A} p(a) \cdot I_p(a) = -\sum_{a \in A} p(a) \cdot \log_2 p(a). \tag{2}$$

The properties of the probabilistic entropy are summarized in many classical works on information theory.

2.2 Fuzzy information measures

In this paper, we interpret a fuzzy set as a *vague information source* (A, μ), where A is an alphabet, and for any $a \in A$, $\mu(a)$ is a value of the membership function (c.f. [9, 1, 4, 7, 5]). The fuzzy entropy $H(A, \mu)$ is defined in several ways. Here, we mention the definition by De Luca and Termini [1], let us denote it

$$H_{\mathrm{LT}}(A, \mu) = -K \cdot \sum_{a \in A} \mu(a) \cdot \log_2 \mu(a), \tag{3}$$

and its more sophisticated analogy suggested by Kolesárová and Vivona in [4], which we denote

$$H_{\mathrm{KV}}(A, \mu) = -K \cdot \sum_{a \in A} \left((\mu(a) \cdot \log_2 \mu(a)) + (1 - \mu(a)) \cdot \log_2(1 - \mu(a))\right), \tag{4}$$

where in both cases K is a positive normalising constant.

When analyzing formulas (3) and (4), we can interpret them as (rather modified) analogies of the arithmetic weighted means of values

$$I_\mu(a) = -\log_2 \mu(a), \quad a \in A. \tag{5}$$

To stress this analogy with [8] we call the values $I_\mu(a)$ the *fuzzy information* of symbol a.

3 Monotonous alternative of fuzzy information

The fuzzy information defined by (5) has interesting advantages but also serious methodological disadvantages. Namely the latter ones were discussed in [7] and partly in [6], too. They regard the observation that the approach represented by (3) and (4) and (5) is not adequate to the elementary paradigm of fuzzy set theory.

Certain attempt to modify the definition of fuzzy-like form was done in [6] and especially [7].

4 General model of fuzzy information measure

Considering a fuzzy information source (A, μ), we aim to suggest some measure of fuzzy information connected with particular symbols $a \in A$. Let us formulate its elementary properties which are rationally demanded. We denote the considered fuzzy information measure by $I_m(a)$, $a \in A$. Then we assume that for $a, b \in A$

$$I_m(a) \geq 0, \tag{6}$$
$$I_m(a) \leq 1, \tag{7}$$
$$I_m(a) = 0 \text{ iff } \mu(a) = 1, \tag{8}$$
$$\text{if } \mu(a) \geq \mu(b) \text{ then } I_m(a) \leq I_m(b). \tag{9}$$

Lemma 1 *The fuzzy information measure $I_\mu(\cdot)$ defined by (5) fulfils (6), (8) and (9).*

Proof The statements follow from the elementary properties of logarithmic function. □

Remark 1 As the alphabet A is assumed to be finite, there evidently always exists real-valued K, where $0 < K \leq 1$, such that $K \cdot I_\mu(a) \leq 1$ for all $a \in A$.

Remark 2 Let us note that even the probabilistic information measure $I_p(\cdot)$ defined by (1) (see [8]) fulfils conditions (6), (8), (9), as well, and for finite alphabet A, it fulfils a statement analogous to previous Remark 1.

4.1 Monotonous information of particular symbol

Our aim is to construct an information measure $I_m(\cdot)$ fulfilling (6), (7), (8), (9), and consequently preserving the structure of fuzzy set theoretical operations.

We call the mapping $I_m \colon A \to [0,1]$ such that for any $a \in A$

$$I_m(a) = 1 - \mu(a), \tag{10}$$

the *monotonous fuzzy information.* It is easy to verify the validity of the following statement.

Theorem 1 *The monotonous fuzzy information $I_m(\cdot)$ defined by (10) fulfils properties (6), (7), (8), and (9).*

Proof The statement of Theorem1 follows from (10), immediately. □

4.2 Extended monotonous information of several symbols

For the probabilistic information [8], let us recollect that for $a, b \in A$, $p(a,b) = p(a) \cdot p(b|a)$ or, in the case of independence, $p(a,b) = p(a) \cdot p(b)$. It justifies the application of the logarithmic function in (1), and the construction of random information I_p as an additive mapping, i.e.,

$$I_p(a,b) = I_p(b) + I_p(a|b) \tag{11}$$

or, in the case of independence,

$$I_p(a,b) = I_p(a) + I_p(b). \tag{12}$$

We can expect that the fuzzy information measure would be rather monotonous than additive, for example.

Namely, it is rational to expect that for any (however defined) measure of information $I(\cdot)$ over A

$$I(a,b) \geq \max\left(I(a), I(b)\right), \quad a, b \in A. \tag{13}$$

Remark 3 Condition (13) is fulfilled for the random information $I_p(\cdot)$ defined by (5), as follows from (11) and (13).

Natural extension of the fuzzy set theoretical representation of uncertainty can be formulated by a fuzzy logical conjunction of the possibilities of symbols a and b. In symbols,

$$\mu(a,b) = \min\left(\mu(a), \mu(b)\right). \tag{14}$$

Then the following statement is evident.

Lemma 2 *If the membership function μ of the fuzzy source (A, μ) is extended from A on A^2 by means of (14) then the fuzzy information measures I_μ and I_m can be extended on A^2, as well (analogously to (5)) and (10), and the extensions $I_\mu(\cdot,\cdot)$, $I_m(\cdot,\cdot)$ fulfil condition (12).*

Proof Due to (5) and (14), condition (13) is evidently fulfilled. □

Lemma 3 *Analogously, under the assumptions of Lemma 2, the extended fuzzy information $I_\mu(\cdot,\cdot)$ fulfils 6), (7) and (9). For finite alphabet A, it fulfils also the corresponding analogy of Remark 1.*

Theorem 2 *The extension of monotonous fuzzy information $I_m(\cdot,\cdot)$ defined by (10) on A^2, fulfils conditions (6), (7), (8) and(9), if the assumptions of Lemma 2 are fulfilled.*

Proof The proof follows from (10) and (14), immediately. □

It is easy to see that extension defined by (13) and (14) need not be limited on the pairs of symbols.

Let (A, μ) be a fuzzy source, let μ^* be an extension of μ on A^* such that for any $n = 1, 2, 3, \ldots$ and any $(a_1, a_2, \ldots, a_n) \in A^n \subset A^*$

$$\mu^*(a_1, \ldots, a_n) = \min\left(\mu(a_1), \mu(a_2), \ldots, \mu(a_n)\right).$$

Then we extend the monotonous vague information measure I_m on the fuzzy source (A^*, μ^*) by means of

$$I_m(a_1, \ldots, a_n) = 1 - \mu^*(a_1, \ldots, a_n). \tag{15}$$

Theorem 3 *Let $a, b, c \in A$, let I_m be defined by (10) and extended by (15) for arbitrary $n = 2, 3, \ldots$ Then*

$$\begin{aligned}
I_m(a, a) &= I_m(a), \\
I_m(a, b) &= I_m(b, a), \\
I_m(a, b, c) &= 1 - \min\left(\mu(a), \mu^*(b, c)\right) = 1 - \min\left(\mu^*(a, b), \mu(c)\right) \\
&= 1 - \min\left(\mu(a), \mu(b), \mu(c)\right).
\end{aligned}$$

Proof Using (13) and (15), it is evident that $\mu(a, a) = \mu(a)$ and $\mu(a, b) = \mu(b, a)$. Hence, the equalities are easily proved. □

5 Two conclusive remarks

The elementary concepts and ideas briefly formulated in the previous sections, can be further developed in several ways. Let us mention, here, at least two of them which appear especially promising.

5.1 Monotonous vague entropy

Analogously to the probabilistic model, the fuzzy information measure of symbols can be extended to the entropy-like concept for entire source. It was done in [7], where the desired type of *fuzzy entropy* is called the *stricly monotonous* one, denoted by $H_{\mathrm{SM}}(A, \mu)$, and defined by

$$H_{\mathrm{SM}}(A, \mu) = 2 \cdot \max\left[\min(I_m(a), 1 - I_m(a)) \colon a \in A\right], \tag{16}$$

which is equivalent to

$$H_{\mathrm{SM}}(A,\mu) = 2 \cdot \max\left[\min(\mu(a), 1-\mu(a))\colon a \in A\right].$$

As shown in [7],

— $H_{\mathrm{SM}}(A,\mu) = 0$ if for all $a \in A$, $\mu(a) \in \{0,1\}$.

— If A, B are finite alphabets and (A,μ), (B,ν) fuzzy sources, where B is a permutation of A, and $\{\nu(b)\colon b \in B\}$ is a permutation of $\{\mu(a)\colon a \in A\}$, then $H_{\mathrm{SM}}(A,\mu) = H_{\mathrm{SM}}(B,\nu)$.

— If $(A,\mu),(A,\nu)$ are fuzzy information sources and for any

$$\mu(a) \le \nu(a) \text{ if } \nu(a) \le \tfrac{1}{2}$$
$$\mu(a) \ge \nu(a) \text{ if } \mu(a) \ge \tfrac{1}{2}$$

then $H_{\mathrm{SM}}(A,\mu) \le H_{\mathrm{SM}}(A,\nu)$.

— If for $(A,\mu),(A,\nu)$ and for all $a \in A$

$$\mu(a) = 1 - \nu(a)$$

then $H_{\mathrm{SM}}(A,\overline{\mu}) = H_{\mathrm{SM}}(A,\nu)$.

— $H_{\mathrm{SM}}(A,\mu) \ge H_{\mathrm{SM}}(A,\nu)$ for all (A,ν) if $\overline{\mu}^*(a) = \frac{1}{2}$ for all $a \in A$.

5.2 Generalization possibilities

This fuzzy approach can be generalized and the fuzzy set theoretical methods can be substituted, e.g., by the t-norms and t-conorms and related tools, eventually by the methods of aggregation operators theory.

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