Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **51**, 2 (2012) 19–25

Stability of Noor Iteration for a General Class of Functions in Banach Spaces

Alfred Olufemi BOSEDE

Department of Mathematics, Lagos State University
Ojo, Lagos State, Nigeria
e-mail: aolubosede@yahoo.co.uk

(Received January 23, 2012)

Abstract

In this paper, we prove the stability of Noor iteration considered in Banach spaces by employing the notion of a general class of functions introduced by Bosede and Rhoades [6]. We also establish similar result on Ishikawa iteration as a special case. Our results improve and unify some of the known stability results in literature.

Key words: stability, Noor and Ishikawa iterations

2000 Mathematics Subject Classification: 47J25, 47H10, 54H25

1 Introduction

Many stability results have been obtained by various authors using different contractive definitions.

Let (E, d) be a complete metric space, $T\colon E \to E$ a selfmap of E; and $F_T = \{p \in E : Tp = p\}$ the set of fixed points of T in E.

For example, Harder and Hicks [10] considered the following concept to obtain various stability results:

Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T in E. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, \ldots \tag{2}$$

Then, the iteration procedure (1) is said to be *T-stable or stable with respect to T* if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

By observing that metric is induced by the norm, (2) becomes

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|, \quad n = 0, 1, 2, \ldots, \tag{3}$$

whenever E is a normed linear space or a Banach space.

If in (1),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, 2, \ldots,$$

then, we have the Picard iteration process. Also, if in (1),

$$f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \ldots,$$

with $\{\alpha_n\}_{n=0}^{\infty}$ a sequence of real numbers in $[0, 1]$, then we have the Mann iteration process.

For any $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tu_n \\ u_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \end{aligned} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1].

For arbitrary $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tq_n \\ q_n &= (1 - \beta_n)x_n + \beta_n Tr_n \\ r_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0, 1]$.

2 Preliminaries

Several researchers in literature including Rhoades [23] and Osilike [20] obtained a lot of stability results for some iteration procedures using various contractive definitions. For example, Osilike [20] considered the following contractive definition: there exist $L \geq 0$, $a \in [0, 1)$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y). \tag{6}$$

Later, Imoru and Olatinwo [12] extended the results of Osilike [20] and proved some stability results for Picard and Mann iteration processes using the following contractive condition: there exist $b \in [0, 1)$ and a monotone increasing function φ: $\Re^+ \to \Re^+$ with $\varphi(0) = 0$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + bd(x, y). \tag{7}$$

Recently, Bosede and Rhoades [6] observed that the process of "generalizing" (6) could continue ad infinitum. As a result of this observation, Bosede and

Rhoades [6] introduced the notion of a general class of functions to prove the stability of Picard and Mann iterations. (For Example, See Bosede and Rhoades [6]).

Our aim in this paper is to prove the stability of Noor iteration for a general class of functions as well as establish similar result on Ishikawa iteration as a special case.

We shall employ the following contractive definition: Let $(E, \|.\|)$ be a Banach space, $T : E \to E$ a selfmap of E, with a fixed point p such that for each $y \in E$ and $0 \leq a < 1$, we have

$$\|p - Ty\| \leq a \|p - y\| . \tag{8}$$

Remark 1 The contractive condition (8) is more general than those considered by Imoru and Olatinwo [12], Osilike [20] and several others in the following sense: By replacing L in (6) with more complicated expressions, the process of "generalizing" (6) could continue ad infinitum. In this paper, we make an obvious assumption implied by (6), and one which renders all generalizations of the form (7) *unnecessary*.

Also, the condition "$\varphi(0) = 0$" usually imposed by Imoru and Olatinwo [12] in the contractive definition (7) is **no longer necessary** in our contraction condition (8) and this is a further improvement to several known stability results in literature.

In the sequel, we shall use the following Lemma which is contained in Berinde [2].

Lemma 1 *Let δ be a real number satisfying $0 \leq \delta < 1$, and $\{\epsilon_n\}$ a positive sequence satisfying $\lim_{n\to\infty} \epsilon_n = 0$. Then, for any positive sequence $\{u_n\}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,$$

it follows that $\lim_{n\to\infty} u_n = 0$.

3 Main results

Theorem 1 *Let $(E, \|.\|)$ be a Banach space, $T: E \to E$ a selfmap of E with a fixed point p, satisfying the contractive condition (8). For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration process defined by (5) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1] such that*

$$0 < \alpha \leq \alpha_n, \quad 0 < \beta \leq \beta_n, \quad 0 < \gamma \leq \beta_n, \quad \text{for all } n. \tag{9}$$

Then, Noor iteration process is T-stable.

Proof Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to p. Suppose also that $\{y_n\}_{n=0}^{\infty} \subset E$ is an arbitrary sequence in E. Define

$$\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Tq_n\| , \quad n = 0, 1, \ldots ,$$

where $q_n = (1-\beta_n)y_n + \beta_n T r_n$ and $r_n = (1-\gamma_n)y_n + \beta_n T y_n$.

Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then, using the contractive condition (8) and the triangle inequality, we shall prove that $\lim_{n\to\infty} y_n = p$ as follows:

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T q_n\| + \|(1-\alpha_n)y_n + \alpha_n T q_n - p\| \\
&= \epsilon_n + \|(1-\alpha_n)y_n + \alpha_n T q_n - ((1-\alpha_n) + \alpha_n)p\| \\
&= \epsilon_n + \|(1-\alpha_n)(y_n - p) + \alpha_n(T q_n - p)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|T q_n - p\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|p - T q_n\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|p - q_n\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|q_n - p\|. \qquad (10)
\end{aligned}$$

For the estimate of $\|q_n - p\|$ in (10), we get

$$\begin{aligned}
\|q_n - p\| &= \|(1-\beta_n)y_n + \beta_n T r_n - p\| \\
&= \|(1-\beta_n)y_n + \beta_n T r_n - ((1-\beta_n) + \beta_n)p\| \\
&= \|(1-\beta_n)(y_n - p) + \beta_n(T r_n - p)\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n \|T r_n - p\| \\
&= (1-\beta_n)\|y_n - p\| + \beta_n \|p - T r_n\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n a \|p - r_n\| \\
&= (1-\beta_n)\|y_n - p\| + \beta_n a \|r_n - p\|. \qquad (11)
\end{aligned}$$

Substitute (11) into (10) gives

$$\|y_{n+1} - p\| \leq \epsilon_n + \Big(1 - (1-a)\alpha_n - \alpha_n\beta_n a\Big)\|y_n - p\| + \alpha_n\beta_n a^2 \|r_n - p\|. \quad (12)$$

For $\|r_n - p\|$ in (12), we have

$$\begin{aligned}
\|r_n - p\| &= \|(1-\gamma_n)y_n + \gamma_n T y_n - p\| \\
&= \|(1-\gamma_n)y_n + \gamma_n T y_n - ((1-\gamma_n) + \gamma_n)p\| \\
&= \|(1-\gamma_n)(y_n - p) + \gamma_n(T y_n - p)\| \\
&\leq (1-\gamma_n)\|y_n - p\| + \gamma_n \|T y_n - p\| \\
&= (1-\gamma_n)\|y_n - p\| + \gamma_n \|p - T y_n\| \\
&\leq (1-\gamma_n)\|y_n - p\| + \gamma_n a \|p - y_n\| \\
&= (1-\gamma_n + \gamma_n a)\|y_n - p\|. \qquad (13)
\end{aligned}$$

Substituting (13) into (12) and using (9), we get

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \epsilon_n + \Big(1 - (1-a)\alpha_n - \alpha_n\beta_n a\Big)\|y_n - p\| \\
&\quad + \alpha_n\beta_n a^2(1 - \gamma_n + \gamma_n a)\|y_n - p\| \\
&= \epsilon_n + \Big(1 - (1-a)\alpha_n - (1-a)\alpha_n\beta_n a - (1-a)\alpha_n\beta_n\gamma_n a^2\Big)\|y_n - p\| \\
&\leq \Big(1 - (1-a)\alpha - (1-a)\alpha\beta a - (1-a)\alpha\beta\gamma a^2\Big)\|y_n - p\| + \epsilon_n. \quad (14)
\end{aligned}$$

Observe that

$$0 \leq \left(1-(1-a)\alpha-(1-a)\alpha\beta a-(1-a)\alpha\beta\gamma a^2\right) < 1.$$

Therefore, taking the limit as $n \to \infty$ of both sides of the inequality (14), and using Lemma 1, we get $\lim_{n\to\infty} \|y_n - p\| = 0$, that is,

$$\lim_{n\to\infty} y_n = p.$$

This completes the proof. □

Theorem 2 *Let $(E, \|.\|)$ be a Banach space, $T\colon E \to E$ a selfmap of E with a fixed point p, satisfying the contractive condition* (8)*. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration process defined by* (4) *converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$ such that*

$$0 < \alpha \leq \alpha_n \quad \text{and} \quad 0 < \beta \leq \beta_n, \quad \text{for all } n. \tag{15}$$

Then, Ishikawa iteration process is T-stable.

Proof Assume that $\{x_n\}_{n=0}^{\infty}$ converges to p. Assume also that $\{y_n\}_{n=0}^{\infty} \subset E$ is an arbitrary sequence in E. Set

$$\epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T u_n\|, \quad n = 0, 1, \ldots,$$

where $u_n = (1-\beta_n)y_n + \beta_n T y_n$. Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n\to\infty} y_n = p$.

Using the contractive condition (8) and the triangle inequality, we have

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T u_n\| \\
&\quad + \|(1-\alpha_n)y_n + \alpha_n T u_n - p\| \\
&= \epsilon_n + \|(1-\alpha_n)y_n + \alpha_n T u_n - ((1-\alpha_n)+\alpha_n)p\| \\
&= \epsilon_n + \|(1-\alpha_n)(y_n - p) + \alpha_n(T u_n - p)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|T u_n - p\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|p - T u_n\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|p - u_n\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|u_n - p\|.
\end{aligned} \tag{16}$$

We estimate $\|u_n - p\|$ in (16) as follows:

$$\begin{aligned}
\|u_n - p\| &= \|(1-\beta_n)y_n + \beta_n T y_n - p\| \\
&= \|(1-\beta_n)y_n + \beta_n T y_n - ((1-\beta_n)+\beta_n)p\| \\
&= \|(1-\beta_n)(y_n - p) + \beta_n(T y_n - p)\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n \|T y_n - p\| \\
&= (1-\beta_n)\|y_n - p\| + \beta_n \|p - T y_n\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n a \|p - y_n\| \\
&= (1-\beta_n + \beta_n a)\|y_n - p\|.
\end{aligned} \tag{17}$$

Substituting (17) into (16) and using (15), we have

$$\begin{aligned}\|y_{n+1}-p\| &\leq \epsilon_n + \Big(1-(1-a)\alpha_n-(1-a)a\alpha_n\beta_n\Big)\|y_n-p\| \\ &\leq \Big(1-(1-a)\alpha-(1-a)a\alpha\beta\Big)\|y_n-p\|+\epsilon_n. \qquad (18)\end{aligned}$$

Since

$$0 \leq \Big(1-(1-a)\alpha-(1-a)a\alpha\beta\Big) < 1,$$

then taking the limit as $n \to \infty$ of both sides of (18), and using Lemma 1, we have

$$\lim_{n\to\infty}\|y_n-p\| = 0,$$

that is,

$$\lim_{n\to\infty} y_n = p.$$

This completes the proof. □

References

[1] Agarwal, R. P., Meehan, M., O'Regan, D.: Fixed Point Theory and Applications. *Cambridge University Press*, Cambridge, 2001.

[2] Berinde, V.: Iterative Approximation of Fixed Points. *Editura Efemeride*, Baia Mare, 2002.

[3] Bosede, A. O.: *Noor iterations associated with Zamfirescu mappings in uniformly convex Banach spaces*. Fasciculi Mathematici **42** (2009), 29–38.

[4] Bosede, A. O.: *Some common fixed point theorems in normed linear spaces*. Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **49**, 1 (2010), 19–26.

[5] Bosede, A. O.: *Strong convergence results for the Jungck–Ishikawa and Jungck-Mann iteration processes*. Bulletin of Mathematical Analysis and Applications **2**, 3 (2010), 65–73.

[6] Bosede, A. O., Rhoades, B. E.: *Stability of Picard and Mann iterations for a general class of functions*. Journal of Advanced Mathematical Studies **3**, 2 (2010), 23–25.

[7] Bosede, A. O., Akinbo, G.: *Some stability theorems associated with A-distance and E-distance in uniform spaces*. Acta Universitatis Apulensis **26** (2011), 121–128.

[8] Ciric, L. B.: *Fixed point theorems in Banach spaces*. Publ. Inst. Math. (Beograd) **47**, 61 (1990), 85–87.

[9] Ciric, L. B.: Fixed Point Theory. Contraction Mapping Principle. *FME Press*, Beograd, 2003.

[10] Harder, A. M., Hicks, T. L.: *Stability results for fixed point iteration procedures*. Math. Japonica **33** (1988), 693–706.

[11] Imoru, C. O., Akinbo, G., Bosede, A. O.: *On the fixed points for weak compatible type and parametrically $\varphi(\epsilon,\delta;a)$-contraction mappings*. Math. Sci. Res. Journal **10**, 10 (2006), 259–267.

[12] Imoru, C. O., Olatinwo, M. O.: *On the stability of Picard and Mann iteration processes*. Carpathian J. Math. **19**, 2 (2003), 155–160.

[13] Imoru, C. O., Olatinwo, M. O., Akinbo, G., Bosede, A. O.: *On a version of the Banach's fixed point theorem*. General Mathematics **16**, 1 (2008), 25–32.

[14] Ishikawa, S.: *Fixed points by a new iteration method*. Proc. Amer. Math. Soc. **44** (1974), 147–150.

[15] Mann, W. R.: *Mean value methods in iterations*. Proc. Amer. Math. Soc. **4** (1953), 506–510.

[16] Noor, M. A.: *General variational inequalities*. Appl. Math. Letters **1** (1988), 119–121.

[17] Noor, M. A.: *New approximations schemes for general variational inequalities*. J. Math. Anal. Appl. **251** (2000), 217–299.

[18] Noor, M. A.: *Some new developments in general variational inequalities*. Appl. Math. Computation **152** (2004), 199–277.

[19] Noor, M. A., Noor, K. I., Rassias, T. M.: *Some aspects of variational inequalities*. J. Comput. Appl. Math. **47** (1993), 493–512.

[20] Osilike, M. O.: *Stability results for fixed point iteration procedures*. J. Nigerian Math. Soc. **14/15** (1995/1996), 17–29.

[21] Rhoades, B. E.: *Fixed point iteration using infinite matrices*. Trans. Amer. Math. Soc. **196** (1974), 161–176.

[22] Rhoades, B. E.: *Comments on two fixed point iteration methods*. J. Math. Anal. Appl. **56**, 2 (1976), 741–750.

[23] Rhoades, B. E.: *Fixed point theorems and stability results for fixed point iteration procedures*. Indian J. Pure Appl. Math. **21**, 1 (1990), 1–9.

[24] Rus, I. A., Petrusel, A., Petrusel, G.: Fixed Point Theory, 1950–2000, Romanian Contributions. *House of the Book of Science*, Cluj-Napoca, 2002.

[25] Zamfirescu, T.: *Fix point theorems in metric spaces*. Arch. Math. **23** (1972), 292–298.

[26] Zeidler, E.: Nonlinear Functional Analysis and its Applications: Fixed Point Theorems. *Springer*, New York, 1986.