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Fixed Point Theorems of the Banach and Krasnosel'skii Type for Mappings on m-tuple Cartesian Product of Banach Algebras and Systems of Generalized Gripenberg's Equations*

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Abstract

In this paper we prove some fixed point theorems of the Banach and Krasnosel'skii type for mappings on the m-tuple Cartesian product of a Banach algebra X over $\mathbb{R}$. Using these theorems existence results for a system of integral equations of the Gripenberg's type are proved. A sufficient condition for the nonexistence of blowing-up solutions of this system of integral equations is also proved.

Key words: fixed point, Banach algebra, integral equation, integro-differential system, epidemic model, blowing-up solution

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G. Gripenberg studied in [7] the integral equation

$$x(t) = k\left(p(t) + \int_0^t A(t-s)x(s)\,ds\right)\left(f(t) + \int_0^t a(t-s)x(s)\,ds\right), \ t \geq 0, \quad (1)$$

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where $k > 0$ is a constant, $p(t)$, $f(t)$, $A(t)$, $a(t)$ are continuous functions. This equation arises in the study of the spread of an infectious disease that does not induce permanent immunity. In the paper [5] the following generalized Gripenberg's equation

$$x(t) = \left(g_1(t) + \int_0^t A_1(t-s)x(s)ds\right) \dots \left(g_p(t) + \int_0^t A_p(t-s)x(s)\, ds\right) \quad (2)$$

is studied. An existence result for this equation is proved there. Its proof is based on an application of the Banach fixed point theorem, where a method of calculation of the constant of contractivity of the corresponding operator defined by the right hand side of (2) is presented. I. M. Olaru [9, 10] proved some existence theorems for the scalar integral equation

$$x(t) = \prod_{i=1}^{m} A_i(x)(t), \quad t \in [a, b], \quad (3)$$

where

$$A_i(x)(t) = g_i(t) + \int_a^t K_i(t, s, x(s))\, ds, \quad t \in [a, b] \quad (4)$$

proved via a weakly Picard operator technique and a technique suggested by E. Brestovanská [5].

In Section 3 we prove an existence theorem for a system of integral equations of the form

$$x_i(t) = \lambda f_i(x) + \mu \prod_{j=1}^{n_i} A_{ij}(x)(t), \quad i = 1, 2, \dots, m, \quad (5)$$

where $n_1, n_2, \dots, n_m, m$ are integers, $\lambda, \mu \in \mathbb{R}$ are parameters and the nonlinear operators $A_{ij}(x), f_i(x)$ will be specified later. This existence result is proved by an application of a fixed point theorem for mappings on Banach algebras which will be formulated and proved in the next section. We also prove a theorem on the nonexistence of blowing-up solutions of the system (5) in a finite $t = \tau$.

1 Fixed point theorem of the Banach type for operators on the m-tuple Cartesian product of a Banach algebra

We shall prove the following fixed point theorem for mappings on the m-tuple Cartesian product of a Banach algebra X over $\mathbb{R}$ which enables us to prove an existence result for a system of integral equations defined in Section 3. We recall that the Banach algebra X is a Banach space over $\mathbb{R}$, where an additional multiplication ab is defined such that $ab \in X$ for all $a, b \in X$ and moreover for all $a, b, c \in X$, $\alpha \in \mathbb{R}$ the following holds: $(ab)c = a(bc)$, $a(b + c) = ab + ac$, $(b + c)a = ba + bc$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$, $\|ab\| \leq \|a\|\|b\|$. In addition, we postulate that there exists an element $e \in X$ such that $ae = ea$ for all $a \in X$ and $\|e\| = 1$ (see e.g. [11]).

Theorem 1 *Let X be a Banach algebra with the multiplication $ab, a, b \in X$ and the norm $|a|_X$. Let X^m be the m-tuple Cartesian product of X with the multiplication $u \star v = (u_1v_1, u_2v_2, \ldots, u_mv_m)$ for $u = (u_1, u_2, \ldots, u_m)$, $v = (v_1, v_2, \ldots, v_m) \in X^m$ and the norm $\|u\| = \max\{|u_1|_X, |u_2|_X, \ldots, |u_m|_X\}$. Let $F\colon X^m \to X^m$, $x \mapsto (F_1(x), F_2(x), \ldots, F_m(x))$, $f\colon X^m \to X^m$, $x \mapsto (f_1(x), f_2(x), \ldots, f_m(x))$ where*

$$F_i(x) = \prod_{j=1}^{n_i} F_{ij}(x),\ i \in \{1, 2, \ldots, m\},$$

n_i is an integer, $F_{ij}\colon X^m \to X$ are mappings with the following properties:

(1) *There is a constant $K > 0$ such that*

$$|F_{ij}(x)|_X \le K$$

for each $(i, j) \in \kappa := \{(l, k)\colon k = 1, 2, \ldots, n_l,\ l = 1, 2, \ldots, m\}$, $x \in X^m$;

(2) *For each $(i, j) \in \kappa$ there exist constants $L_{ij} > 0$ such that*

$$|F_{ij}(x) - F_{ij}(y)|_X \le L_{ij}\|x - y\|$$

for all $x, y \in X^m$;

(3) *There exists a constant $\omega > 0$ such that*

$$\|f(x) - f(y)\| \le \omega\|x - y\|$$

for all $x, y \in X^m$;

(4) *$|\lambda|\omega + |\mu|k < 1$, where*

$$k := \max\{K^{n_i-1}(L_{i1} + L_{i2} + \ldots L_{in_i})\colon i = 1, 2, \ldots, m\}.$$

Then the mapping $H := \lambda f + \mu F$ has a unique fixed point $u^* = (u_1^*, u_2^*, \ldots, u_m^*) \in X^m$, i.e. the system

$$u_1^* = \lambda f_1(x) + \mu \prod_{j=1}^{n_1} F_{1j}(u^*), u_2^* = \lambda f_2(x) + \mu \prod_{j=1}^{n_2} F_{2j}(u^*), \ldots, u_m^* = \lambda f_m(x) + \mu \prod_{j=1}^{n_m} F_{mj}(u^*). \tag{6}$$

has a unique solution u^*. Moreover, for any $u_0 \in X^m$ the sequence $\{u_k\}_{k=0}^{\infty}$ with $u_{k+1} = (\lambda f + \mu F)(u_k)$ converges to u^*.

Proof Let $i \in \{1, 2, \ldots, m\}, x, y \in X^m$. Then

$$\begin{aligned} |F_i(x) - F_i(y)|_X &\le |F_{i1}(x)F_{i2}(x)\ldots F_{in_i}(x) - F_{i1}(y)F_{i2}(y)\ldots F_{in_i}(y)|_X \\ &+ |F_{i1}(x)F_{i2}(x)\ldots F_{in_i}(x) - F_{i1}(y)F_{i2}(x)F_{i3}(x)\ldots F_{in_i}(x)|_X \\ &+ |F_{i1}(y)F_{i2}(x)F_{i3}(x)\ldots F_{in_i}(x) - F_{i1}(y)F_{i2}(y)F_{i3}(x)\ldots F_{in_i}(x)|_X \\ &+ \cdots + |F_{i1}(y)F_{i2}(y)F_{i3}(y)\ldots F_{in_i-1}(y)F_{in_i}(x) - F_{i1}(y)F_{i2}(y)F_{i3}(y)\ldots F_{in_i}(y)|_X \\ &\le L_{i1}K^{n_i-1}\|x-y\| + \ldots L_{in_i}K^{n_i-1}\|x-y\| = K^{n_i-1}(L_{i1} + L_{i2} + \cdots + L_{in_i})\|x-y\|. \end{aligned}$$

This yields

$$\|F(x) - F(y)\| = \max\{|F_i(x) - F_i(y)|_X : i = 1, 2, \ldots, m\} \le k\|x - y\|.$$

If $H := \lambda f + \mu F$ then $\|H(x) - H(y)\| \le (\lambda\omega + \mu k)\|x - y\|$ for all $x, y \in X^m$. By the assumption (4) $|\lambda|\omega + |\mu|k < 1$ and thus the assertion of the theorem follows from the Banach fixed point theorem. □

Remark 1 This fixed point theorem can be applied e.g. in the case if X is the set of real numbers with norm given by absolute value, or the set of all real or complex $n \times n$-matrices with a sub-multiplicative norm, or $\mathbb{R}^n$, or C^n both with the norm $\|x\|_n = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, $x = (x_1, x_2, \ldots, x_n)$ and the multiplication defined as

$$(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) = (x_1y_2, x_2y_2, \ldots, x_ny_n).$$

We shall apply Theorem 1 to the case of a system of equations on m-tuple X^m of the Banach algebra $X = C[a, b]$ of all continuous real-valued functions on a compact interval $I = [a, b]$ with the norm $|f|_C = \max_{t\in I} |f(t)|$. Then the norm on the Banach algebra X^m is $\|g\|_{C,m} := \max\{|g_1|_C, |g_2|_C, \ldots, |g_m|_C\}$, $g = (g_1, g_2, \ldots, g_m) \in X^m$.

2 Fixed point theorem of the Krasnosel'skii type for operators on the m-tuple Cartesian product of a Banach algebra

In this short section we formulate a fixed point theorem of Krasnosel'skii type using the property of the mapping F from Theorem 1.

First let us recall the Krasnosel'skii fixed point theorem (see e.g. [8]).

Theorem 2 *Let M be a nonempty closed convex subset of a Banach space X and A, B be two mappings from M to X such that*

(a) A is compact and continuous;

(b) B is a contraction;

(c) $A(x) + B(y) \in M$ for all $x, y \in M$.

Then the mapping $A + B$ has at least one fixed point in M.

Theorem 3 *Let X be a Banach algebra with the multiplication $ab, a, b \in X$ and the norm $|a|_X$. Let X^m be the m-tuple Cartesian product of X with the multiplication $u \star v = (u_1v_1, u_2v_2, \ldots, u_mv_m)$ for $u = (u_1, u_2, \ldots, u_m)$, $v = (v_1, v_2, \ldots, v_m) \in X^m$ and the norm $\|u\| = \max\{|u_1|_X, |u_2|_X, \ldots, |u_m|_X\}$ and $K \subset X^m$ is a nonempty closed convex subset of the Banach space X^m. Let*

$$F\colon X^m \to X^m, x \mapsto (F_1(x), F_2(x), \ldots, F_m(x))$$

be a mapping satisfying the conditions (1), (2) *from Theorem 1 with $k < 1$, where the number k is defined in the condition (4) of this theorem. Let*

$$G\colon X^m \to X^m,\ x \mapsto (G_1(x), G_2(x), \ldots, G_m(x))$$

be a mapping satisfying the conditions

(a) G is compact and continuous;

(b) $F(x) + G(y) \in K$ for all $x, y \in K$.

Then the mapping $F + G$ has a fixed point in K.

Since the number $k < 1$ is the constant of the contractivity of the mapping F, the conditions (a), (b) implies that the assumptions of the Krasnosel'skii theorem with $A = G, B = F$ are satisfied.

In the next section we prove some existence results for systems of of integral equations which are nonlinear perturbation of systems of integral equations of the Gripenberg's type.

3 Existence results for a system of integral equations which is a perturbation of a system of the Gripenberg's type

First we apply Theorem 1 to the following system of equations

$$x_i(t) = \lambda \left(c_i + \int_a^t H_i(t, s, x(s))\, ds \right) + \mu \prod_{j=1}^{n_i} \left(g_{ij}(t) + \int_a^t R_{ij}(t, s, x(s))\, ds \right), \quad (7)$$

$t \in I = [a, b]$, $i = 1, 2, \ldots, m$, $x = (x_1, x_2, \ldots, x_m)$, $\lambda, \mu \in \mathbb{R}$, n_i, $i = 1, 2, \ldots, m$, are integers, $g_{ij}(t)$ are continuous functions defined on a compact interval $I = [a, b] \subset \mathbb{R}$, $\eta_i(t, x)$ are continuous functions on $I \times \mathbb{R}^m$ and $H_i, R_{ij} \colon I \times I \times \mathbb{R}^m \to \mathbb{R}$ are continuous functions.

The system (7) is a generalization of the Gripenberg equation (1), the equation (2) studied by the author in [5] and also the equation (3) studied by I. M. Olaru in [9, 10].

A special system of the form (7) can be obtained from the following integro-differential system

$$x_i(t) = \lambda y_{i1}(t) + \mu \left(g_i(t) + \int_a^t R_i(t, s, x(s))\, ds \right) y_{i2}(t) \ldots y_{im}(t)$$

$$\dot{y}_{i1}(t) = f_{i1}(t, x(t)), \ \ldots, \ \dot{y}_{im}(t) = f_{im}(t, x(t))$$
$$y_{i1}(a) = y_{i1}^0, \ y_{i2}(a) = y_{i2}^0, \ \ldots, \ y_{im}(a) = y_{im}^0,$$

$i = 1, 2, \ldots, m$, $x = (x_1, x_2, \ldots, x_m)$, $t \in I = [a, b]$. This system can be written in the integral form

$$x_i(t) = \lambda \left(y_{i1}^0 + \int_a^t f_{i1}(s, x(s)) ds \right)$$

$$+ \mu \left(g_i(t) + \int_a^t R_i(t, s, x(s)) ds \right) \prod_{j=2}^{m} \left(y_{ij}^0 + \int_a^t f_{ij}(s, x(s)) ds \right), \ i = 1, 2, \ldots, m.$$

We can write the system (7) as the following equation on the Banach algebra X^m with $X = C[a,b]$:

$$x = \lambda G(x) + \mu F(x), \quad x \in X^m, \tag{8}$$

where $G\colon X^m \to X^m$, $G = (G_1, G_2, \dots, G_m)$,

$$G_i(x)(t) := c_i + \int_a^t H_i(t,s,x(s))\,ds, \quad t \in I,$$

$F\colon X^m \to X^m$, $F = (F_1, F_2, \dots, F_m)$,

$$F_i(x)(t) := \prod_{j=1}^{n_i} \left(g_{ij}(t) + \int_a^t R_{ij}(t,s,x(s))\,ds \right), \quad t \in I = [a,b],\ i = 1,2,\dots,m.$$

By a solution of the system (7) on the interval I we mean a continuous mapping $x = (x_1, x_2, \dots, x_m)\colon I \to \mathbb{R}$ such that $x(t) = (x_1(t) \dots, x_m(t))$ satisfies the equation (7) for each $t \in \{1, 2, \dots, m\}$.

Theorem 4 *Let m and $n_i, i = 1,2,\dots,m$ be integers,*

$$\kappa = \{(l,k)\colon k = 1,2,\dots,n_l,\ l = 1,2,\dots,m\},$$

$g_{ij}(t)$, $(i,j) \in \kappa$ be continuous functions defined on a compact interval $I = [a,b]$ and $c_i \in \mathbb{R}$, $H_i, R_{ij}, f_i\colon I \times I \times \mathbb{R}^m \to \mathbb{R}$, $(i,j) \in \kappa$ be continuous functions satisfying the following conditions:

(1) There are positive continuous functions $K_{ij}(t,s)$ on $I \times I$ such that

$$|R_{ij}(t,s,u)| \le K_{ij}(t,s) \tag{9}$$

for all $(i,j) \in \kappa$, $(t,s) \in I \times I$, $u \in \mathbb{R}^m$.

(2) There are positive continuous functions $l_{ij}(t,s)$, $(i,j) \in \kappa$ on $I \times I$ such that

$$|R_{ij}(t,s,u) - R_{ij}(t,s,v)| \le l_{ij}(t,s)\|u - v\|_m \tag{10}$$

for all $(i,j) \in \kappa$, $(t,s) \in I \times I$, $u, v \in \mathbb{R}^m$.

(3) There exist continuous nonnegative functions $h_i\colon I \times I \to \mathbb{R}$, $i = 1,2,\dots,m$ such that

$$|H_i(t,s,u) - H_i(t,s,v)| \le h_i(t,s)\|u - v\|_m,$$
$$(t,s) \in I \times I,\ u, v \in \mathbb{R}^m,\ i = 1,2,\dots,m. \tag{11}$$

Then there exist numbers $d_1 > 0$, $d_2 > 0$ such that for all $\lambda, \mu \in \mathbb{R}$ with $|\lambda| \le d_1$, $|\mu| \le d_2$ the equation (8) has a unique solution in the Banach algebra X^m.

Proof We will apply Theorem 1. Let $X = C[a,b]$ with the norm $|x|_C$, X^m with the norm $\|g\|_{C^m} = \max\{|g_1|_C, |g_2|_C, \dots, |g_m|_C\}$,

$$g = (g_1, g_2, \dots, g_m) \in X^m,\ F = (F_1, F_2, \dots, F_m)\colon X^m \to X^m,$$

where

$$F_i(x) = \prod_{j=1}^{n_i} F_{ij}(x), \quad i = 1, 2, \dots, m,$$

$$F_{ij}(x)(t) = g_{ij}(t) + \int_a^t R_{ij}(t, s, x(s))\, ds, \quad x \in X^m, \ (i, j) \in \kappa, \ t \in I.$$

The existence of a solution of the system (7) is equivalent to the existence of a fixed point of the mapping $H = \lambda G + \mu F$, where

$$G(x)(t) := (G_1(x)(t), G_2(x)(t), \dots, G_m(x)(t)),$$

$$G_i(x)(t) = c_i + \int_a^t H_i(t, s, x(s))\, ds, \quad t \in I, \ x \in X^m.$$

From the property (1) we have

$$|F_{ij}(x)|_C = \max_{t \in I} |F_{ij}(x)(t)| \le K := A + K_0, \tag{12}$$

where $K = A + K_0$,

$$K_0 := \max \left\{ \int_a^t K_{ij}(t, s)\, ds \colon t \in I, \, (i, j) \in \kappa \right\},$$

$$A = \max\{|g_{ij}(t)| \colon (i, j) \in \kappa, \, t \in I\},$$

$$L_{ij} = \max \left\{ \int_a^t l_{ij}(t, s)\, ds \colon t \in I, \, (i, j) \in \kappa \right\}.$$

If $x, y \in X^m$ then the property (2) yields

$$|F_{ij}(x) - F_{ij}(y)|_C = \max_{t \in I} |F_{ij}(x)(t) - F_{ij}(y)(t)|$$

$$= \max_{t \in I} \int_a^t |R_{ij}(t, s, x(s)) - R_{ij}(t, s, y(s))| ds$$

$$\le \max_{t \in I} \int_a^t l_{ij}(t, s) \|x(s) - y(s)\|_m ds \le L_{ij} \|x - y\|_{C^m}.$$

Therefore $\|F(x) - F(y)\|_{C^m} \le k \|x - y\|_{C^m}$ for all $x, y \in X^m$, where

$$k = \max\{K^{m_i - 1} \, (L_{i1} + L_{i2} + \dots + L_{in_i}) : i = 1, 2, \dots, m\}, \tag{13}$$

If $x, y \in X^m$ then

$$|G_i(x) - G_i(y)|_C = \max_{t \in I} |G_i(x)(t) - G_i(y)(t)|$$

$$= \max_{t \in I} \int_a^t |H_i(t, s, x(s)) - H_i(t, s, y(s))|\, ds$$

$$\le \max_{t \in I} \int_a^t h_i(t, s) \|x(s) - y(s)\|_m \, ds \le \omega_i \|x - y\|_{C^m},$$

where

$$\omega_i = \max\left\{\int_a^t h_i(t,s)\,ds,\ i = 1,2,\dots m\right\}.$$

This yields

$$\|G(x) - G(y)\|_{C^m} \le \omega\|x - y\|_{C^m},$$

where $\omega = \max\{\omega_i : i = 1,2,\dots,m\}$.

If we take $d_1 > 0, d_2 > 0$ such that $d_1\omega + d_2 k < 1$, and $H := \lambda G + \mu F$, then $\|\lambda H(x) - H(y)\|_{C^m} \le (d_1\omega + d_2 k)\|x - y\|_{C^m}$ for all $x, y \in X^m$. The assertion of Theorem 4 follows from the Banach fixed point theorem. The proof is completed. □

The following theorem is a direct consequence of Theorem 1 and Theorem 2.

Theorem 5 *Let the assumptions of Theorem 1 be satisfied, $M \subset X^m$ be a nonempty closed convex subset and $G\colon M \to X^m$, $X = C[a,b]$ and let the following conditions are satisfied:*

(a) $G\colon M \to X^m$ is a compact and continuous mapping;

(b) $F\colon M \to X^m$ be the mapping from Theorem 1 with $k < 1$;

(c) $F(x) + G(y) \in M$ for all $x, y \in M$.

Then the mapping $F + G$ has at least one fixed point in M.

We will apply this theorem to the following system of integral equations.

$$x_i(t) = \lambda G_i(x)(t) + \mu F_i(x)(t), \quad i = 1,2,\dots,m,\ x = (x_1, x_2, \dots, x_m), \tag{14}$$

where $\lambda, \mu \in \mathbb{R}$ are parameters,

$$F = (F_1, F_2, \dots, F_m)\colon X^m \to X^m,$$

$$F_i(x) = \prod_{j=1}^{n_i} F_{ij}(x), \quad i = 1,2,\dots,m,$$

$$F_{ij}(x)(t) = g_{ij}(t) + \int_a^t R_{ij}(t,s,x(s))\,ds, \quad x \in X^m,\ (i,j) \in \kappa,\ t \in I.$$

and the mapping $G = (G_1, G_2, \dots, G_m)\colon X^m \to X^m$ is defined by

$$G(x)(t) := \int_a^t H(t,s,x(s))\,ds, \quad t \in [a,b],$$

where $H = (H_1, H_2, \dots, H_m)\colon I \times I \times \mathbb{R}^m \to \mathbb{R}^m$, $I = [a,b]$. We can write this system as the following equation on the Banach algebra X^m, where $X = C[0,b]$.

$$x(t) = \lambda G(x)(t) + \mu F(x)(t). \tag{15}$$

Theorem 6 *Assume the following hypotheses:*

(1) The mapping $F\colon X^m \to X^m$,

$$F_i(x) = \prod_{j=1}^{n_i} F_{ij}(x), \quad i = 1, 2, \ldots, m,$$

$$F_{ij}(x)(t) = g_{ij}(t) + \int_a^t R_{ij}(t, s, x(s))\, ds, \quad x \in X^m,\ (i,j) \in \kappa,\ t \in I$$

and let the conditions (1), (2) of Theorem 4 be satisfied;

(2) The mapping $H = (H_1, H_2, \ldots, H_m)\colon I \times I \times \mathbb{R}^m \to \mathbb{R}^m$ *is continuous;*

(3) Let the positive numbers c_1, c_2, r *be given such that* $c_2 k < 1$, *where* k *is defined in the condition* (4) *of Theorem 1 (see (13)), and* $c_1 \mathbb{M}(b-a) + c_2 K \le r$, *where the number* K *is defined in the proof of Theorem 4 (see (12)) and*

$$\mathbb{M} := \max\{\|H(t,s,u)\|_m \colon (t,s,u) \in Q_r\},$$
$$Q_r := \{(t,s,u) \in I \times I \times \mathbb{R}^m \colon \|u\|_m \le r\}.$$

Then for any $\lambda, \mu \in \mathbb{R}$ *with* $|\lambda| \le c_1, |\mu| \le c_2$ *the mapping* $\lambda G(x) + \mu F(x)$ *has at least one fixed point in* $B_{C^m}(0,r) := \{v \in X^m \colon \|v\|_{C^m} \le r\}$, *i.e. the system* (14) *has at least one solution.*

Proof Denote $B = B_{C^m}(0,r)$ the closed ball in X^m with the center at the origin and the radius r. Define the operator

$$P_{\lambda,\mu}\colon B \to X^m,\ P_{\lambda,\mu}(x) = \lambda G(x) + \mu F(x),\ x \in B.$$

Let $x \in B$, $t \in I$, $P_{\lambda,\mu} = (P_1, P_2, \ldots, P_m)$, $i \in \{1, 2, \ldots, m\}$. Then

$$|P_i(x)(t)| \le |\lambda||G_i(x)(t)| + |\mu||F_i(x)(t)|$$
$$\le |\lambda|\mathbb{M}(b-a) + |\mu|K \le c_1\mathbb{M}(b-a) + c_2 K \le r$$

for all λ, μ with $|\lambda| \le c_2$, $|\mu| \le c_2$ and all $t \in I$. This means that $P_{\lambda,\mu}(B) \subset B$ for all λ, μ with $|\lambda| \le c_2$, $|\mu| \le c_2$. Therefore the operator $P_{\lambda,\mu}(x) = \lambda G(x) + \mu F(x)$ satisfies the condition (3) of the Krasnosel'skii theorem for all such values of the parameters λ, μ. From [11] (see pages 40, 41) it follows that the integral operator λG is compact and continuous. Since k is the Lipschitz constant of the mapping F the assumption $c_2 k < 1$ implies that the operator μF is contractive for all μ with $|\mu| \le c_2$. Thus we have proved that also the conditions (1) and (2) of the Krasnosl'skii theorem are fulfilled and therefore the mapping $\lambda F + \mu G$ has at least one fixed point in B. □

4 On the nonexistence of blowing-up solutions

We have proved the existence theorem for the system (7) on a compact interval $[a,b]$. Now let us study the system (7) on the interval $[a,\infty)$. We shall prove a result on the nonexistence of blowing-up solutions for this system, where by a blowing-up solution of (7) we mean a solution $x(t)$ for which there is a point $\tau > a$ such that it exists on the interval $[a, \tau)$ and $lim_{t\to\tau^-}\|x(t)\|_m = \infty$. The following lemma can be found in [4].

Theorem 7 *Let m and n_i, $i = 1, 2, \ldots, m$ be integers,*

$$\kappa = \{(l, k) \colon k = 1, 2, \ldots, n_l,\ l = 1, 2, \ldots, m\},$$

$g_{ij}(t), (i, j) \in \kappa$, be continuous functions defined on the interval $I_a = [a, \infty)$, $H_i(t, s, u)$, $i = 1, 2, \ldots, m$ are continuous functions on the set $I_a \times I_a \times \mathbb{R}^m \to \mathbb{R}^m$ and $R_{ij} \colon I_a \times I_a \times \mathbb{R}^m \to \mathbb{R}$, $(i, j) \in \kappa$ be continuous functions satisfying the following conditions:

(1) For each $(i, j) \in \kappa$ there exists a continuous, positive and nondecreasing function $\gamma_{ij}(u)$, $u \geq 0$ and a continuous positive function $r_{ij}(t, s)$ on $I_a \times I_a$ that

$$|R_{ij}(t, s, u)| \leq r_{ij}(t, s)\gamma_{ij}(\|u\|_m)$$

for all $(t, s) \in I_a \times I_a$, $u \in \mathbb{R}^m$.

(2) There exist continuous positive functions $h_i \colon I_a \times I_a \to \mathbb{R}$, $i = i, 2, \ldots, m$ and a continuous nondecreasing positive function $\omega \colon [0, \infty) \to \mathbb{R}$ such that

$$|H_i(t, s, u)| \leq h_i(t, s)\omega(\|u\|_m), \quad (t, s, u) \in I_a \times I_a \times \mathbb{R}.$$

(3) Let

$$\gamma(u) = \max\{\gamma_{ij}(u) \colon (i, j) \in \kappa\}, \quad u \geq 0,$$

$$H(u) = \max\{\gamma(u)^{n_1}, \gamma(u)^{n_2}, \ldots, \gamma(u)^{n_m}\}, \quad \Psi(w) = \omega(w) + H(w), \quad w \geq 0$$

and assume

$$\int_{u_0}^{\infty} \frac{d\sigma}{\omega(\sigma) + H(\sigma)} = \infty, \quad u_0 > 0.$$

Then the system (7) *does not have a blowing-up solution.*

Proof Let $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))$ be a solution of the system (7), continuous on the interval $[a, \tau), a < \tau < \infty$ with $\lim_{t\to\tau^-} \|x(t)\|_m = \infty$. Using the property (1)–(3) we can estimate $|x_i(t)|$ as follows:

$$\begin{aligned}
|x_i(t)| \leq |\lambda||c_i| + |\lambda| \int_a^t h_i(t, s)\omega(\|u(s)\|_m)\, ds \\
+ |\mu| \prod_{j=1}^{n_i} \left(|g_{ij}(t)| + \int_a^t r_{ij}(t, s)\gamma_{ij}(\|x(s)\|_m)\, ds \right) \\
\leq M_\tau + N_\tau \int_a^t \omega(\|u(\tau)\|_m|)d\tau + \mu| \left(K_\tau + L_\tau \int_a^t \gamma_{ij}(\|x(s)\|_m) ds \right)^{n_i},
\end{aligned}$$

where $M_\tau = |\lambda| \max\{|c_1|, |c_2|, \dots, |c_m|\}$,

$$N_\tau = |\lambda|(\tau - a) \max\{h_i(t,s) \colon t \in [a,\tau],\ i = 1,2,\dots,m\},$$
$$K_\tau = \max\{|g_{ij}(t)| \colon (i,j) \in \kappa,\ t \in [a,\tau]\},$$
$$L_\tau = \max\{|r_{ij}(t,s)| \colon (i,j) \in \kappa,\ (t,s) \in [a,\tau] \times [a,\tau]\}.$$

If $n_i > 1$ then as a consequence of the Hölder inequality we obtain the inequality

$$\left(\int_a^t \gamma(\|x(s)\|_m)\, ds\right)^{n_i} \le (t-a)^{n_i - 1} \int_a^t \gamma_{ij}^{n_i}(\|x(s)\|_m)\, ds.$$

If $n_i = 1$ then obviously this inequality is valid. Using this inequality and the elementary inequality $(\alpha + \beta)^{n_i} \le 2^{n_i - 1}(\alpha^{n_i} + \beta^{n_i})$, where $\alpha, \beta \ge 0$, we obtain the estimate

$$\begin{aligned} |x_i(t)| &\le M_\tau + N_\tau \int_a^t \omega(\|u(\tau)\|_m|)\, d\tau \\ &+ |\mu| \left(2^{n_i - 1}[K_\tau^{n_i} + L_\tau^{n_i} \int_a^t \gamma_{ij}^{n_i}(\|x(s)\|_m)\, ds]\right) \\ &\le M_\tau + N_\tau \int_a^t \omega(\|u(\tau)\|_m|) d\tau \\ &+ 2^{n_i - 1} K_\tau^{n_i} |\mu| + 2^{n_i - 1} L_\tau^{n_i} \tau^{n_i - 1} \int_a^t \gamma_{ij}^{n_i}(\|x(s)\|_m)\, ds, \end{aligned}$$

i.e.

$$|x_i(t)| \le P_i + Q_i (\tau - a)^{n_i - 1} \int_a^t \gamma_{ij}^{n_i}(\|x(s)\|_m)\, ds,$$

where

$$P_i = M_\tau + 2^{n_i - 1} K_\tau^{n_i} |\mu|, \quad Q_i = 2^{n_i - 1} L_\tau^{n_i}.$$

From this inequality we obtain

$$\|x(t)\|_m = \max_{1 \le i \le m} |x_i(t)| \le P + B \int_a^t [\omega(\|u(s)\|_m) + H(\|x(s)\|_m)]\, ds,$$

where

$$H(u) = \max\{\gamma(u)^{n_1}, \gamma(u)^{n_2}, \dots, \gamma(u)^{n_m}\},$$
$$\gamma(u) = \max\{\gamma_{ij}(u) \colon (i,j) \in \kappa\}, \quad u \ge 0,$$
$$P = \max\{P_1, P_2, \dots, P_m\}, \quad B = \max\{N_\tau, bQ\}, \quad Q = \max\{Q_1, Q_2, \dots, Q_m\},$$
$$b = \max\{(\tau - a)^{n_1 - 1}, (\tau - a)^{n_2 - 1}, \dots, (\tau - a)^{n_m - 1}\}.$$

Now using the Bihari inequality (see [4]) we obtain

$$\Omega(\|x(t)\|_m) = \int_{u_0}^{\|x(t)\|_m} \frac{d\sigma}{\omega(\sigma) + H(\sigma)} \le \Omega(P) + B(t-a), \quad u_0 = \|x(a)\|_m,\ t \ge a. \tag{16}$$

From the condition (3) we have

$$\lim_{t\to\infty}\Omega(\|x(t)\|)=\lim_{\tau\to\tau^-}\int_{u_0}^{\|x(t)\|}\frac{d\sigma}{\omega(\sigma)+H(\sigma)}=\infty.$$

However this is the contradiction because the limit of the right-hand side of the inequality (16) as $t\to\tau^-$ is finite. □

Example 1 Let the assumption (1) of Theorem 7 is satisfied, where $n_1=n_2=\dots n_m=2$ and $\gamma_{ij}(u)=\gamma(u)=\ln(2+u)$, $u\geq a=u_0\geq 0$. Then the function $H(u)$ from the condition (2) has the form $H(u)=[\ln(2+u)]^2$ and we have

$$\int_a^\infty\frac{d\sigma}{H(\sigma)}=\int_a^\infty\frac{d\sigma}{[\ln(2+\sigma)]^2}\geq\int_a^\infty\frac{d\sigma}{(2+\sigma)\ln(2+\sigma))}=\infty. \tag{17}$$

Thus the assumptions of Theorem 7 are satisfied. Therefore the system (7) has no solution blowing-up solution.

5 Concluding remarks

Let us mention papers in which some fixed point theorems in Banach algebras are applied to integral equations. These fixed point theorems differ from our Theorem 1. In the paper [2] a functional-integral equation of the form

$$x(t)=f\left(t,\int_0^t v(t,s,x(s))\,ds,x(\alpha(t))\right)g\left(t,\int_0^a u(t,s,x(s)\,ds,x(\beta(t))\right),\ t\in[0,a] \tag{18}$$

is solved in the Banach algebra $C[0,a]$ by applying a fixed point theorem for the product $S=P.T$ of operators in Banach algebra E proved in the paper [3]. It is assumed there that the operators P,T satisfy the Darbo condition, i. e. $\mu(SX)\leq k\mu(X)$ for each $X\in\mathcal{M}_E$ - a family of all nonempty and bounded subsets of E, where $\mu\colon\mathcal{M}_E\to\mathbb{R}_+$ is a regular measure of noncompactness and k is a positive constant with respect to μ. It is also assumed there that P and T transforms continuously a nonempty, bounded, convex and closed set $\Omega\subset C[a,b]$ into $C[a,b]$ in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded and that there are positive constants k_1,k_2 such that $\|P(\Omega)\|k_2+\|T(\Omega)\|k_1<1$. Under these conditions the operator S has at least one fixed point in Ω. Our theorem Theorem 1 can be applied almost directly to systems of equations of the form

$$x_i(t)=f_i\left(t,\int_a^t v(t,s,x(s)\,ds\right)g_i\left(t,\int_a^b u(t,s,x(s))\,ds\right),\ i=1,2,\dots,n, \tag{19}$$

where $x=(x_1,x_2,\dots,x_n)$,

$$f_i\left(t,\int_a^t v(t,s,x(s))\,ds\right):=g_{i1}(t)+\int_a^t R_{i1}(t,s,x(s))\,ds, \tag{20}$$

$$g_i\left(t, \int_a^b u(t,s,x(s))\,ds\right) := g_{i2}(t) + \int_a^b R_{i2}(t,s,x(s))\,ds, \;\; a \le t \le b. \tag{21}$$

This type of equations differs from those in the system (7) by the upper bound of the integral in the second terms which is constant. However an analogous result to Theorem 4 can be formulated and its proof is almost the same as the proof of our theorem.

We remark that another type of fixed point theorem in Banach algebras (Furi–Pera type fixed point theorems) are proved in [6] and they are applied in the proof of an existence theorem for some scalar functional-integral equations which are in an multiplicative form as the equation (19).

References

[1] Abdeldaim, A.: *On some new Gronwall-Bellman-Ou-Iang type integral inequalities to study certain epidemic model.* J. Integral Equations Appl. (2011), (to appear).

[2] Banas, J., Sadarangani, K.: *Solutions of some functional-integral equations in Banach Algebra.* Math. Comput. Modelling **38** (2003), 245–250.

[3] Banas, J., Lacko, M.: *Fixed points of the product of operators in Banach algebra.* Panamer. Math. J. **12** (2002), 101–109.

[4] Bihari, I.: *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations.* Acta Mathematica Hungarica **7**, 1 (1956), 81–94.

[5] Brestovanská, E.: *Qualitative behaviour of an integral equation related to some epidemic model.* Demonstratio Math. **36**, 3 (2003), 604–609.

[6] Djebali, S., Hammache, K.: *Furi-Pera fixed point theorems in Banach algebra with applications.* Acta Univ. Palacki. Olomouc, Fac. Rer. Nat., Math. **47** (2008), 55–75.

[7] Gripenberg, G.: *On some epidmic models.* Quart. Appl. Math. **39** (1981), 317–327.

[8] Krasnosel'skii, M. A.: *Two remarks on the method of succesive approximations.* Uspechi Mat. Nauk **10** (1955), 123–127.

[9] Olaru, I. M.: *An integral equation via weakly Picard operators.* Fixed Point Theory **11**, 1 (2010), 97–106.

[10] Olaru, I. M.: *Generalization of an integral equation related to some epidemic models.* Carpatian J. Math. **26** (2010), 0–4.

[11] Zeidler, E.: Applied Functional Analysis: Applications to Mathematical Physics. Applied Mathematical Sciences, Vol. 108, *Springer-Verlag*, New York, 1995.