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On Existence and Asymptotic Properties of Kneser Solutions to Singular Second Order ODE*

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Abstract

We investigate an asymptotic behaviour of damped non-oscillatory solutions of the initial value problem with a time singularity $(p(t)u'(t))' + p(t)f(u(t)) = 0$, $u(0) = u_0$, $u'(0) = 0$ on the unbounded domain $[0, \infty)$. Function f is locally Lipschitz continuous on $\mathbb{R}$ and has at least three zeros $L_0 < 0$, 0 and $L > 0$. The initial value $u_0 \in (L_0, L) \setminus \{0\}$. Function p is continuous on $[0, \infty)$, has a positive continuous derivative on $(0, \infty)$ and $p(0) = 0$. Asymptotic formulas for damped non-oscillatory solutions and their first derivatives are derived under some additional assumptions. Further, we provide conditions for functions p and f, which guarantee the existence of Kneser solutions.

Key words: singular ordinary differential equation of the second order, time singularities, unbounded domain, asymptotic properties, Kneser solutions, damped solutions, non-oscillatory solutions

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1 Introduction

There exists an extensive literature which is devoted to a qualitative analysis of solutions of the nonlinear equation

$$(p(t)u')' + q(t)f(u(t)) = 0, \tag{1.1}$$

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where p, q and f are continuous on their domains. A lot of results are obtained in the case that (1.1) is the Emden–Fowler equation, that is $p \equiv 1$ and $f(x) = |x|^\gamma \operatorname{sgn} x$, $\gamma > 0$, $\gamma \neq 1$. See e.g. [12], [14], [15], [16], [20] and [29]. Equation (1.1) with a nonconstant p and a more general f has been investigated in [4], [5], [13], [17] and [30], while quasilinear equations can be found in [2], [10], [11] and [31]. The nonlinearity f in all these papers has globally monotonous behaviour which is characterized by the assumption $xf(x) > 0$ for all $x \neq 0$.

In our paper we investigate equation (1.1) with $p = q$ and $p(0) = 0$. We would like to stress that for our results the condition $xf(x) > 0$ need not be fulfilled for all $x \neq 0$ (see condition (1.5)). Since the case $\int_0^1 \frac{\mathrm{d}t}{p(t)} = \infty$ is also considered here, the differential operator in equation (1.1) can have a singularity at $t = 0$. This is a fundamental difference from the papers cited above. It is shown in [18] that such type of singular operators appears in models of various practical problems, for example in hydrodynamics [3], [9], [28], in population genetics [7], [8], in the homogeneous nucleation theory [1], in relativistic cosmology [19], in the nonlinear field theory [6].

In particular, we study here the equation

$$(p(t)u'(t))' + p(t)f\left(u(t)\right) = 0 \tag{1.2}$$

on the half-line $[0, \infty)$ under the following assumptions:

$$L_0 < 0 < L,\ f(L_0) = f(0) = f(L) = 0, \tag{1.3}$$

$$f \in \mathrm{Lip}_{loc}(\mathbb{R}), \tag{1.4}$$

$$xf(x) > 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{1.5}$$

$$F(L_0) > F(L), \text{ where } F(x) = \int_0^x f(z)\,\mathrm{d}z, \tag{1.6}$$

$$p \in C[0,\infty) \cap C^1(0,\infty),\ p(0) = 0, \tag{1.7}$$

$$p' > 0 \text{ on } (0,\infty),\ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{1.8}$$

In the whole paper we will assume that conditions (1.3)–(1.8) hold. Assumption (1.6) implies

$$\exists \overline{B} \in (L_0, L)\colon F(\overline{B}) = F(L). \tag{1.9}$$

We are interested in a solution u which meets the following definition.

Definition 1.1 A function $u \in C^1[0,\infty) \cap C^2(0,\infty)$ which satisfies equation (1.2) for all $t \in (0,\infty)$ is called a *solution* of (1.2).

Definition 1.2 Let u be a solution of equation (1.2) and let L be of (1.3). If

$$\sup\{u(t)\colon t \in [0,\infty)\} < L, \tag{1.10}$$

then u is called a *damped* solution of (1.2).

Definition 1.3 A solution u of equation (1.2) is called *oscillatory* solution if it has an unbounded set of zeros. If a solution u of equation (1.2) has a finite set of zeros, then it is called a *non-oscillatory* solution. Further, u is called *eventually positive* (*eventually negative*), if there exists $t_0 \ge 0$ such that $u(t) > 0$ ($u(t) < 0$) for $t \in [t_0, \infty)$.

Remark 1.4 By (1.4), solutions of equation (1.2) cannot vanish together with their first derivative at a point from $(0, \infty)$. Therefore each non-oscillatory solution of equation (1.2) is either eventually positive or eventually negative.

Let us consider the initial conditions

$$u(0) = u_0, \ u'(0) = 0, \tag{1.11}$$

where $u_0 \in (L_0, L) \setminus \{0\}$. In [24], under assumptions (1.3)–(1.8), it was proved that for each $u_0 \in [\overline{B}, L)$ there exists a (unique) solution of problem (1.2), (1.11) which is damped (see Theorem 2.3 in [24]). On the other hand, if $u_0 \in (L_0, \overline{B})$, then the corresponding (unique) solution of problem (1.2), (1.11) need not be damped (see [25] for more details). Further in [24], it was shown that both damped oscillatory solutions and damped non-oscillatory solutions of equation (1.2) can exist. We demonstrate it in the next example.

Example 1.5 Consider the equation

$$\left(t^3 u'(t)\right)' + t^3 f\left(u(t)\right) = 0, \tag{1.12}$$

where

$$f(x) = \begin{cases} -12 - 2x & \text{for } x < -2, \\ x^3 & \text{for } x \in [-2, 1], \\ 2 - x & \text{for } x > 1. \end{cases}$$

Here $L_0 = -6$, $L = 2$, $\overline{B} = -\sqrt[4]{3}$, $p(t) = t^3$. We can check by a direct computation that for $u_0 \in [-2, 1]$, problem (1.12), (1.11) has a solution

$$u(t) = \frac{8u_0}{8 + u_0^2 t^2}, \quad t \in [0, \infty).$$

If $u_0 \in (0, 1]$, the solution u is positive and decreasing in $[0, \infty)$, and hence it is damped non-oscillatory. Similarly, we see that if $u_0 \in [-2, 0)$, then u is negative and increasing on $[0, \infty)$, and so it is also damped and non-oscillatory. On the other hand, numerical simulations give damped oscillatory solutions provided $u_0 \in (1, 2)$.

All possible types of solutions to equation (1.2) (homoclinic, escape, damped) have been studied in [22]–[27]. In our paper we investigate in more details damped non-oscillatory solutions of (1.2) starting at a singular point $t = 0$ and we provide an interval for starting values u_0 giving Kneser solutions (see Theorems 3.4 and 3.5). We specify their behaviour by asymptotic formulas (see Theorems 2.1, 2.2 and 2.3). The first information about asymptotic behaviour of such solutions is given in the following lemma.

Lemma 1.6 [24, Lemma 2.6] *Let u be a damped non-oscillatory solution of problem* (1.2), (1.11) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then*

$$\lim_{t\to\infty} u(t) = 0, \ \lim_{t\to\infty} u'(t) = 0. \tag{1.13}$$

In order to derive asymptotic formulas, we will need moreover the assumptions

$$\limsup_{t\to\infty} \frac{tp'(t)}{p(t)} \in [1, \infty), \tag{1.14}$$

$$\exists r > 1\colon \ \liminf_{x\to 0} \frac{|f(x)|}{|x|^r} > 0, \ \limsup_{x\to 0} \frac{|f(x)|}{|x|^r} < \infty. \tag{1.15}$$

Remark 1.7 Let us show that the condition

$$\limsup_{t\to\infty} \frac{tp'(t)}{p(t)} \in (0, 1) \tag{1.16}$$

need not be considered here. We can see that if (1.16) holds, then there exists $\lambda \in (0, 1)$ and a sufficiently large $T > 0$ such that

$$0 < \frac{tp'(t)}{p(t)} < \lambda, \quad t > T.$$

Therefore

$$\int_T^t \frac{p'(s)}{p(s)} \, \mathrm{d}s < \lambda \int_T^t \frac{1}{s} \, \mathrm{d}s, \quad t > T,$$

and

$$\frac{1}{p(t)} > \frac{(T)^\lambda}{p(T)} t^{-\lambda}, \quad t > T.$$

Since $\int_1^\infty t^{-\lambda} \, \mathrm{d}t = \infty$, we get

$$\int_1^\infty \frac{\mathrm{d}t}{p(t)} = \infty. \tag{1.17}$$

By Theorem 4.6 and Remark 4.3 in [27], condition (1.17) implies that each damped solution of (1.2), (1.11) with $u_0 \in (L_0, 0) \cup (0, L)$ is oscillatory.

Remark 1.8 Let (1.15) hold with $r = 1$. If moreover

$$\limsup_{t\to\infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty$$

and $p \in C^2(0, \infty)$, then by Theorem 2.10 in [24], each damped solution of (1.2), (1.11) with $u_0 \in (L_0, 0) \cup (0, L)$ is oscillatory. In addition, if we take $r \in (0, 1)$ in (1.15), then $\lim_{x\to 0^+} \frac{f(x)}{x^r} = c > 0$ and

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{f(x)}{x^r} x^{r-1} = c \lim_{x\to 0^+} x^{r-1} = \infty.$$

Therefore f does not fulfil (1.4) at $x = 0$. Hence, it is reasonable to have $r > 1$ in (1.15).

2 Asymptotic formulas

In this section, asymptotic formulas for a damped non-oscillatory solution of problem (1.2), (1.11) are derived in Theorem 2.1. Further, asymptotic formulas for its first derivative are proved in Theorem 2.2, and under stronger assumptions in Theorem 2.3.

Theorem 2.1 *Let* (1.14) *and* (1.15) *hold. Let u be a damped non-oscillatory solution of problem* (1.2), (1.11) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then*

$$\limsup_{t\to\infty} t^{\frac{2}{r-1}} |u(t)| < \infty. \tag{2.1}$$

Proof Let u be a damped non-oscillatory solution. Due to Remark 1.4 solution u is either eventually positive or eventually negative. If u is eventually positive, then there exists $t_0 \geq 0$ such that $u(t) > 0$ for $t \in [t_0, \infty)$. By (1.13), we can find $t_1 \geq t_0$ such that $u(t_1) \in (0, L), u'(t_1) \leq 0$ and $u > 0$ on $[t_1, \infty)$. Then conditions (1.5), (1.7) and (1.8) imply $(pu')' < 0$ on $[t_1, \infty)$. Consequently, pu' is decreasing and hence $u'(t) < 0$ on (t_1, ∞). Similarly, if u is eventually negative, then $u < 0$ for $t \in [t_0, \infty)$, and there exists $t_1 \geq t_0$ such that $u'(t) > 0$ on (t_1, ∞). By (1.15), there exist $\alpha, \beta > 0$ and $\delta > 0$ such that

$$\alpha < \frac{|f(x)|}{|x|^r} < \beta, \quad x \in (0, \delta).$$

Condition (1.13) yields $a \geq t_1$ such that

$$0 < |u(t)| < \delta, \quad t \geq a.$$

Hence we obtain

$$\alpha |u(t)|^r < |f(u(t))| < \beta |u(t)|^r, \quad t \geq a. \tag{2.2}$$

Further we integrate equation (1.2) from a to $t \geq a$

$$p(t)u'(t) - p(a)u'(a) + \int_a^t p(s) f(u(s))\, \mathrm{d}s = 0.$$

Since $u(a)u'(a) < 0$, we get

$$p(t)|u'(t)| > \int_a^t p(s)|f(u(s))|\, \mathrm{d}s, \quad t > a.$$

Monotonous behaviour of $u(t)$ for $t > a$ and inequality (2.2) yield

$$p(t)|u'(t)| > \alpha |u(t)|^r \int_a^t p(s)\, \mathrm{d}s.$$

Therefore

$$\frac{|u'(t)|}{\alpha |u(t)|^r} > \frac{1}{p(t)} \int_a^t p(s)\, \mathrm{d}s, \quad t > a. \tag{2.3}$$

Due to (1.14), there exists $\lambda \in (1,\infty)$ and a sufficiently large $T \geq a$ such that

$$\frac{tp'(t)}{p(t)} < \lambda, \quad t > T,$$

and

$$\int_T^t sp'(s)\,\mathrm{d}s < \lambda \int_T^t p(s)\,\mathrm{d}s, \quad t > T. \tag{2.4}$$

Integrating by parts we obtain

$$\int_T^t sp'(s)\,\mathrm{d}s = tp(t) - Tp(T) - \int_T^t p(s)\,\mathrm{d}s.$$

Therefore, by (2.4), for $t > T$,

$$\begin{aligned}
tp(t) - Tp(T) - \int_T^t p(s)\,\mathrm{d}s &< \lambda \int_T^t p(s)\,\mathrm{d}s,\\
tp(t) - Tp(T) &< (\lambda+1) \int_T^t p(s)\,\mathrm{d}s,\\
t - T\frac{p(T)}{p(t)} &< \frac{\lambda+1}{p(t)} \int_T^t p(s)\,\mathrm{d}s,\\
\frac{t-T}{\lambda+1} &< \frac{1}{p(t)} \int_T^t p(s)\,\mathrm{d}s.
\end{aligned}$$

Hence, according to (2.3), we obtain

$$\frac{|u'(t)|}{\alpha|u(t)|^r} > \frac{1}{p(t)} \int_a^t p(s)\,\mathrm{d}s \geq \frac{1}{p(t)} \int_T^t p(s)\,\mathrm{d}s > \frac{t-T}{\lambda+1}, \quad t > T. \tag{2.5}$$

We integrate (2.5) from T to $t > T$ and get

$$\int_T^t \frac{|u'(s)|}{\alpha|u(s)|^r}\,\mathrm{d}s > \frac{1}{\lambda+1} \int_T^t (s-T)\,\mathrm{d}s = \frac{(t-T)^2}{2(\lambda+1)}.$$

Therefore the following estimate holds for $t > T$

$$\frac{1}{\alpha(r-1)} \left(\frac{1}{|u(t)|^{r-1}} - \frac{1}{|u(T)|^{r-1}} \right) > \frac{(t-T)^2}{2(\lambda+1)}.$$

Consequently,

$$\frac{1}{|u(t)|^{r-1}} > \frac{\alpha(r-1)(t-T)^2}{2(\lambda+1)} + \frac{1}{|u(T)|^{r-1}} > \Lambda(t-T)^2, \quad t > T,$$

where $\Lambda = \frac{\alpha(r-1)}{2(\lambda+1)}$. Further, it holds for $t > T$

$$\begin{aligned}
0 &< |u(t)| < \left(\Lambda(t-T)^2\right)^{-1/(r-1)},\\
0 &< |u(t)| < t^{-2/(r-1)} \left(\Lambda(1-T/t)^2\right)^{-1/(r-1)},\\
0 &< t^{2/(r-1)}|u(t)| < \left(\Lambda(1-T/t)^2\right)^{-1/(r-1)}.
\end{aligned}$$

Letting $t \to \infty$ we get

$$\lim_{t\to\infty} \left(\Lambda(1 - T/t)^2\right)^{-1/(r-1)} = \Lambda^{-1/(r-1)} < \infty.$$

This limit implies that formula (2.1) holds. □

The previous result is essential for asymptotic formulas for first derivatives of damped non-oscillatory solutions of problem (1.2), (1.11) as you can see in the proof of following theorem.

Theorem 2.2 *Let all assumptions of Theorem 2.1 be fulfilled. Denote*

$$\lambda = \limsup_{t\to\infty} \frac{tp'(t)}{p(t)}. \tag{2.6}$$

I. *If* $\lambda \in [1, \frac{r+1}{r-1})$, *then*

$$\limsup_{t\to\infty} p(t)|u'(t)| < \infty. \tag{2.7}$$

II. *If* $\lambda \geq \frac{r+1}{r-1}$, *then for any* $\lambda_0 > \lambda$

$$\limsup_{t\to\infty} p(t)t^{\frac{r+1}{r-1}-\lambda_0}|u'(t)| < \infty. \tag{2.8}$$

Proof Let u be damped non-oscillatory and let $t_0 \leq t_1 \leq a$ be the points from the proof of Theorem 2.1. Then $uu' < 0$ on $[a, \infty)$ and (2.2) holds. Choose $\lambda_0 > \lambda$. Due to (2.1) and (2.6), we can find $c > 0$ and a sufficiently large $T \geq a$ such that

$$0 < t^{2/(r-1)}|u(t)| < c, \ 0 < \frac{tp'(t)}{p(t)} < \lambda_0, \quad t > T. \tag{2.9}$$

First, integrate equation (1.2) over (T, t) and put $A_1 = p(T)|u'(T)|$. Then, by (2.2),

$$0 < p(t)|u'(t)| = A_1 + \int_T^t p(s)|f(u(s))|\,\mathrm{d}s < A_1 + \beta \int_T^t p(s)|u(s)|^r\,\mathrm{d}s, \quad t > T.$$

Now, integrate the second inequality in (2.9) over (T, t). As in Remark 1.7 we deduce that

$$p(t) < \frac{p(T)}{T^{\lambda_0}} t^{\lambda_0}, \quad t > T.$$

To summarize it, we get

$$\begin{aligned} 0 \,<\, p(t)|u'(t)| &< A_1 + \beta \frac{p(T)}{T^{\lambda_0}} \int_T^t s^{\lambda_0}|u(s)|^r\,\mathrm{d}s \\ &= A_1 + \beta \frac{p(T)}{T^{\lambda_0}} \int_T^t s^{\lambda_0 - 2r/(r-1)} \left(s^{2/(r-1)}|u(s)|\right)^r \mathrm{d}s, \quad t > T. \end{aligned} \tag{2.10}$$

Put $A_2 = \frac{\beta p(T)}{T^{\lambda_0}}$ and $\mu = \lambda_0 - \frac{2r}{r-1}$. Then, due to the first inequality in (2.9), inequality (2.10) yields

$$0 < p(t)|u'(t)| < A_1 + A_2 c^r \int_T^t s^\mu \, \mathrm{d}s, \quad t > T. \tag{2.11}$$

I. Let $\lambda \in \left[1, \frac{r+1}{r-1}\right)$. Then we choose $\lambda_0 \in \left(\lambda, \frac{r+1}{r-1}\right)$ and get $\mu = \lambda_0 - \frac{2r}{r-1} < -1$. Therefore

$$\lim_{t\to\infty} \int_T^t s^\mu \, \mathrm{d}s < \infty,$$

which together with (2.11) gives (2.7).

II. Let $\lambda \geq \frac{r+1}{r-1}$. Then we choose $\lambda_0 > \lambda$, which implies $\mu = \lambda_0 - \frac{2r}{r-1} > -1$. Therefore

$$\lim_{t\to\infty} t^{-\mu-1} \int_T^t s^\mu \, \mathrm{d}s = \lim_{t\to\infty} t^{-\mu-1} \frac{t^{\mu+1} - T^{\mu+1}}{\mu+1} < \infty,$$

which together with (2.11) gives

$$\limsup_{t\to\infty} p(t) t^{-\mu-1} |u'(t)| < \infty.$$

Since $\mu + 1 = \lambda_0 - \frac{2r}{r-1} + 1 = \lambda_0 - \frac{r+1}{r-1}$, (2.8) holds. $\square$

If we replace assumption (1.14) in Theorem 2.2 by

$$\lim_{t\to\infty} \frac{p'(t)}{t^{n-2}} \in (0,\infty), \text{ for some } n \in (2,\infty), \tag{2.12}$$

we get more precise formulas, as it is seen in the next theorem. Let us note that if (2.12) holds with $n \in (1,2]$, then (1.17) is satisfied. According to Remark 1.7 this case does not concern non-oscillatory solutions.

Theorem 2.3 *Let* (1.15) *and* (2.12) *hold. Let u be a damped non-oscillatory solution of problem* (1.2), (1.11) *with $u_0 \in (L_0, 0) \cup (0, L)$.*

I. *If $n \in \left(2, \frac{2r}{r-1}\right)$, then*

$$\limsup_{t\to\infty} t^{n-1} |u'(t)| < \infty. \tag{2.13}$$

II. *If $n = \frac{2r}{r-1}$, then*

$$\limsup_{t\to\infty} t^{\frac{r+1}{r-1}} \frac{1}{\ln t} |u'(t)| < \infty. \tag{2.14}$$

III. *If $n > \frac{2r}{r-1}$, then*

$$\limsup_{t\to\infty} t^{\frac{r+1}{r-1}} |u'(t)| < \infty. \tag{2.15}$$

Proof By the l'Hôpital rule, assumption (2.12) implies the existence of the limit

$$\lim_{t\to\infty}(n-1)\frac{p(t)}{t^{n-1}} = \lim_{t\to\infty}\frac{p'(t)}{t^{n-2}} \in (0,\infty). \tag{2.16}$$

Consequently condition (1.14) is fulfilled, that is

$$\lim_{t\to\infty}\frac{tp'(t)}{p(t)} = \lim_{t\to\infty}\frac{p'(t)}{t^{n-2}}\frac{t^{n-1}}{p(t)(n-1)}(n-1) = n-1.$$

Therefore we can apply the results of Theorems 2.1 and 2.2. Due to (2.16), there exists $0<\gamma_0<\gamma$ and a sufficiently large $T>0$ such that

$$\gamma_0 t^{n-1} < p(t) < \gamma t^{n-1}, \quad t>T. \tag{2.17}$$

Using (1.2), (2.2), (2.9) and (2.17), we can modify the proof of Theorem 2.2 and obtain the following estimates for a damped non-oscillatory solution u.

$$\begin{aligned} 0 &< p(t)|u'(t)| < A_1 + \beta\gamma\int_T^t s^{n-1}|u(s)|^r\,\mathrm{d}s \\ &= A_1 + \beta\gamma\int_T^t s^{n-1-2r/(r-1)}\left(s^{2/(r-1)}|u(s)|\right)^r\,\mathrm{d}s \\ &< A_1 + \beta\gamma c^r\int_T^t s^{n-1-2r/(r-1)}\,\mathrm{d}s \\ &= A_1 + A_2\int_T^t s^{n-1-2r/(r-1)}\,\mathrm{d}s, \quad t>T, \end{aligned} \tag{2.18}$$

where $A_1 = p(T)|u'(T)|,\ A_2 = \beta\gamma c^r$. Hence, (2.17) and (2.18) give

$$0 < \gamma_0 t^{n-1}|u'(t)| < A_1 + A_2\int_T^t s^{n-1-2r/(r-1)}\,\mathrm{d}s, \quad t>T. \tag{2.19}$$

I. Let $n\in(2,\frac{2r}{r-1})$. Then

$$\lim_{t\to\infty}\int_T^t s^{n-1-2r/(r-1)}\,\mathrm{d}s < \infty.$$

Therefore, by (2.19), we get (2.13).

II. Let $n=\frac{2r}{r-1}$. Then

$$\lim_{t\to\infty}\frac{1}{\ln t}\int_T^t\frac{\mathrm{d}s}{s} < \infty,$$

which together with (2.19) yields (2.14).

III. Let $n>\frac{2r}{r-1}$. Then, by (2.19),

$$\begin{aligned} &\limsup_{t\to\infty} t^{\frac{r+1}{r-1}}|u'(t)| \\ &\le \lim_{t\to\infty}\frac{1}{\gamma_0}\left(A_1 t^{\frac{2r}{r-1}-n} + \frac{A_2}{n-\frac{2r}{r-1}}\left(1-\left(\frac{T}{t}\right)^{\frac{n-2r}{r-1}}\right)\right) = \frac{A_2}{\gamma_0(n-\frac{2r}{r-1})}. \end{aligned}$$

Consequently, (2.15) is proved. □

3 Kneser solutions

The main purpose of this section is to prove the existence of Kneser solutions of equation (1.2), which are defined as follows.

Definition 3.1 A solution u of equation (1.2) is called a *Kneser solution* if there exists $t_0 > 0$ such that

$$u(t)u'(t) < 0 \text{ for } t \in [t_0, \infty). \tag{3.1}$$

First, we show a connection between damped non-oscillatory solutions (see Definition 1.1 and 1.3) and Kneser solutions.

Theorem 3.2 *Let u be a damped non-oscillatory solution of problem* (1.2), (1.11) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is a Kneser solution of equation* (1.2).

Proof According to Remark 1.4, u is either eventually positive or eventually negative. Let u be eventually positive. Then there exists $a > 0$ such that

$$0 < u(t) < L, \quad t \geq a. \tag{3.2}$$

By Lemma 1.6, u fulfils (1.13) and hence there exists $t_0 \geq a$ such that

$$u'(t_0) \leq 0. \tag{3.3}$$

Applying (1.5), (1.7), (1.8) and (3.2) to equation (1.2), we get $(p(t)u'(t))' < 0$ for $t \geq a$, which together with (3.3) yields

$$u'(t) < 0, \quad t \in (t_0, \infty). \tag{3.4}$$

Consequently, by virtue of Definition 3.1 and inequalities (3.2), (3.4), u is a Kneser solution of equation (1.2). If u is eventually negative, we argue similarly. □

In order to prove *sufficient* conditions for the existence of Kneser solutions of equation (1.2), we will need the next identities of Pohozhaev type.

Lemma 3.3 *Let u be a solution of equation* (1.2). *Then u fulfils*

$$p(t)u(t)u'(t) = \int_0^t p(s)u'^2(s)\,\mathrm{d}s - \int_0^t p(s)f\left(u(s)\right)u(s)\,\mathrm{d}s, \quad t > 0, \tag{3.5}$$

$$\begin{aligned} P(t)\left(\frac{u'^2(t)}{2} + F\left(u(t)\right)\right) &= \int_0^t p(s)F\left(u(s)\right)\,\mathrm{d}s \\ &\quad - \int_0^t \left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right)p(s)u'^2(s)\,\mathrm{d}s, \quad t > 0, \end{aligned} \tag{3.6}$$

where $P(t) = \int_0^t p(s)\,\mathrm{d}s$.

Proof Consider a solution u of equation (1.2). To derive equality (3.5), we use equation (1.2) in the form

$$p(t)u''(t) + p'(t)u'(t) + p(t)f(u(t)) = 0, \quad t > 0. \tag{3.7}$$

Then, multiplying equation (3.7) by the solution u, we have

$$p(t)u''(t)u(t) + p'(t)u'(t)u(t) + p(t)f(u(t))\,u(t) = 0, \quad t > 0. \tag{3.8}$$

Using (3.8) together with the equality

$$(p(t)u'(t)u(t))' = p'(t)u'(t)u(t) + p(t)u''(t)u(t) + p(t)u'^2(t), \quad t > 0,$$

we get

$$(p(t)u'(t)u(t))' = p(t)u'^2(t) - p(t)f(u(t))\,u(t), \quad t > 0. \tag{3.9}$$

We integrate (3.9) over $(0, t)$ and obtain (3.5).

Further, we multiply equation (3.7) by $\frac{P(t)u'(t)}{p(t)}$ and get

$$P(t)u''(t)u'(t) + \frac{P(t)p'(t)}{p(t)}u'^2(t) + P(t)f(u(t))\,u'(t) = 0, \quad t > 0. \tag{3.10}$$

According to the two following equalities

$$\begin{aligned} \left(P(t)u'^2(t)\right)' &= p(t)u'^2(t) + 2P(t)u'(t)u''(t), \quad t > 0,\\ (P(t)F(u(t)))' &= p(t)F(u(t)) + P(t)f(u(t))\,u'(t), \quad t > 0, \end{aligned}$$

and due to (3.10), we deduce

$$\begin{aligned} &\left(P(t)\frac{u'^2(t)}{2} + P(t)F(u(t))\right)' \\ &= \frac{p(t)}{2}u'^2(t) - \frac{P(t)p'(t)}{p(t)}u'^2(t) + p(t)F(u(t)) \\ &= -\left(\frac{P(t)p'(t)}{p^2(t)} - \frac{1}{2}\right)p(t)u'^2(t) + p(t)F(u(t)), \quad t > 0. \end{aligned} \tag{3.11}$$

Integrating (3.11) over $(0, t)$ we obtain (3.6). □

The existence of a Kneser solution is guaranteed by two following theorems.

Theorem 3.4 *(On the existence of Kneser solutions I.)*
Assume that there exist $c > \frac{1}{2}$ and $A_0 \in (0, L)$ such that the following inequalities

$$\frac{p'(t)P(t)}{p^2(t)} \geq c, \quad t \in (0, \infty), \tag{3.12}$$

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \quad x \in (0, A_0] \tag{3.13}$$

hold. Then for each $u_0 \in (0, A_0]$ there exists a unique solution u of problem (1.2), (1.11)*. The solution u is damped, fulfils* (1.13) *and*

$$u(t) > 0,\ u'(t) < 0, \quad t \in (0, \infty). \tag{3.14}$$

Proof By Theorem 2.3 in [24] there exists a unique solution u of problem (1.2), (1.11) with $u_0 \in (0, A_0]$. Suppose, on the contrary, that there exists $t_0 > 0$ such that $u(t) > 0$ on $[0, t_0)$ and $u(t_0) = 0$. Due to (1.4), (1.5) and equation (1.2), the inequality $u'(t_0) < 0$ holds and

$$0 < u(t) < A_0,\ u'(t) < 0,\ t \in (0, t_0). \tag{3.15}$$

Now, we use equality (3.5) for $t = t_0$

$$p(t_0)u(t_0)u'(t_0) = \int_0^{t_0} p(s)u'^2(s)\,\mathrm{d}s - \int_0^{t_0} p(s)f\left(u(s)\right)u(s)\,\mathrm{d}s.$$

Since $u(t_0) = 0$, we get

$$\int_0^{t_0} p(s)u'^2(s)\,\mathrm{d}s = \int_0^{t_0} p(s)f\left(u(s)\right)u(s)\,\mathrm{d}s. \tag{3.16}$$

Due to (3.6), where $t = t_0$, it holds

$$\begin{aligned} 0 \;&<\; P(t_0)\left(\frac{u'^2(t_0)}{2} + F\left(u(t_0)\right)\right) \\ &= \int_0^{t_0} p(s)F\left(u(s)\right)\,\mathrm{d}s - \int_0^{t_0}\left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right)p(s)u'^2(s)\,\mathrm{d}s. \end{aligned}$$

According to (3.16) and (3.12) we get

$$\begin{aligned} \int_0^{t_0} p(s)F\left(u(s)\right)\,\mathrm{d}s &> \int_0^{t_0}\left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right)p(s)u'^2(s)\,\mathrm{d}s \\ &\geq \left(c - \frac{1}{2}\right)\int_0^{t_0} p(s)u'^2(s)\,\mathrm{d}s = \frac{2c-1}{2}\int_0^{t_0} p(s)f\left(u(s)\right)u(s)\,\mathrm{d}s. \end{aligned}$$

This yields

$$\int_0^{t_0} p(s)F\left(u(s)\right)\left(\frac{2}{2c-1} - \frac{f\left(u(s)\right)u(s)}{F\left(u(s)\right)}\right)\mathrm{d}s > 0. \tag{3.17}$$

Furthermore,

$$\frac{f\left(u(s)\right)u(s)}{F\left(u(s)\right)} \geq \frac{2}{2c-1}, \quad s \in (0, t_0),$$

is satisfied according to (3.13) and (3.15). Therefore

$$\left(\frac{2}{2c-1} - \frac{f\left(u(s)\right)u(s)}{F\left(u(s)\right)}\right) \leq 0, \quad s \in (0, t_0),$$

contrary to (3.17). The obtained contradiction implies $u(t) > 0$ on $[0, \infty)$. Due to equation (1.2) and (1.5), we get $u'(t) < 0$ for $t \in (0, \infty)$. Hence (3.14) is valid. Since $A_0 \in (0, L)$, u is damped and Lemma 1.6 gives (1.13). □

A dual theorem for initial values from a left neighbourhood of zero is proved with similar arguments.

Theorem 3.5 *(On the existence of Kneser solutions II.)*
Let condition (3.12) *hold with a constant* $c > \frac{1}{2}$ *and assume that there exists* $B_0 \in (L_0, 0)$ *such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \quad x \in [B_0, 0) \tag{3.18}$$

is satisfied. Then for each $u_0 \in [B_0, 0)$ *there exists a unique solution* u *of problem* (1.2), (1.11)*. The solution* u *is damped, fulfils* (1.13) *and*

$$u(t) < 0, \ u'(t) > 0, \quad t \in (0, \infty). \tag{3.19}$$

Proof By Theorem 2.3 in [24] there exists a unique solution u of problem (1.2), (1.11) with $u_0 \in [B_0, 0)$. In the contradiction with (3.19), we suppose the existence of $t_0 > 0$ such that $u(t) < 0$ on $[0, t_0)$ and $u(t_0) = 0$. By virtue of (1.4), (1.5) and equation (1.2), the inequality $u'(t_0) > 0$ holds and

$$B_0 < u(t) < 0, \ u'(t) > 0, \quad t \in (0, t_0). \tag{3.20}$$

Using the identities of Pohozhaev type (3.5), (3.6) and assumptions (3.12), (3.18) and (3.20) as in the proof of Theorem 3.4, we obtain a contradiction which implies that t_0 cannot exist. Therefore (3.19) holds. This implies that u is damped and, by Lemma 1.6, assertion (1.13) is valid. □

Corollary 3.6 *Let* u *be a solution of equation* (1.2) *and let assumption* (3.13) *or* (3.18) *be fulfilled. If the function* p *satisfies condition* (3.12) *only on* $(0, t_0]$ *then, according to the proof of Theorem 3.4 or Theorem 3.5, each solution* u *of problem* (1.2), (1.11) *with* $u_0 \in (0, A_0]$ *or* $u_0 \in [B_0, 0)$ *has no zero on* $(0, t_0]$.

Example 3.7 Let $L_0 < B_0 < 0 < A_0 < L, \ \alpha > 1, \ r > 1$. Consider problem (1.2), (1.11), where

$$p(t) \, = t^{\alpha}, \ t \in [0, \infty), \tag{3.21}$$

$$f(x) \, = \begin{cases} \dfrac{|B_0|^r (L_0 - x)}{B_0 - L_0} & \text{for} \ \ x \in [L_0, B_0), \\ |x|^r \operatorname{sgn} x & \text{for} \ \ x \in [B_0, A_0], \\ \dfrac{A_0^r (x - L)}{A_0 - L} & \text{for} \ \ x \in (A_0, L], \end{cases} \tag{3.22}$$

and f is arbitrary Lipschitz continuous otherwhere and fulfils $F(L_0) > F(L)$. To ensure the existence of Kneser solutions, we apply Theorems 3.4 and 3.5. We see that p satisfies condition (3.12), because

$$\frac{p'(t)P(t)}{p^2(t)} = \frac{\alpha}{\alpha + 1} = c > \frac{1}{2}, \quad t \in (0, \infty).$$

Since

$$\frac{xf(x)}{F(x)} = r + 1 \quad \text{for } x \in [B_0, 0) \cup (0, A_0],$$

conditions (3.13) and (3.18) are reduced to a simple inequality

$$r + 1 \geq \frac{2}{2c - 1}. \tag{3.23}$$

Lower bounds $\frac{2}{2c-1}$ corresponding some values of α are given in the presented table.

α	2	3	4	5	6	7	8	9	10	11
$\frac{2}{2c-1}$	6.00	4.00	3.33	3.00	2.80	2.67	2.57	2.50	2.44	2.40

For instance, put $\alpha = 3$. Then $c = \frac{3}{4}$ and $\frac{2}{2c-1} = 4$. Therefore, due to (3.23), if $r \geq 3$ we can apply Theorems 3.4 and 3.5. As a result, for each $u_0 \in [B_0, 0) \cup (0, A_0]$, problem (1.2), (1.11), where p and f are given by (3.21) and (3.22), respectively, has a damped Kneser solution u. In addition, the function p satisfies (1.14) and (2.12) with $n = 4$ and the function f fulfils (1.15). According to Theorem 2.1, the solution u has the asymptotic formula

$$\limsup_{t \to \infty} t^{\frac{2}{r-1}} |u(t)| < \infty.$$

The inequality $r \geq 3$ yields $\frac{2r}{r-1} \leq 3$ and hence

$$4 = n > \frac{2r}{r - 1}.$$

Thus, an asymptotic formula for u' has the form

$$\limsup_{t \to \infty} t^{\frac{r+1}{r-1}} |u'(t)| < \infty,$$

due to Theorem 2.3.

Example 3.8 Consider problem (1.2), (1.11) where

$$p(t) = t^4 + t^3, \quad t \in [0, \infty), \tag{3.24}$$

$$f(x) = \begin{cases} -\frac{x+3}{2} & \text{for } x < -1 \\ x^3 & \text{for } x \in [-1, 1] \\ 2 - x & \text{for } x > 1. \end{cases} \tag{3.25}$$

We verify that assumptions (3.12), (3.13) and (3.18) are satisfied. For $t > 0$ it holds

$$\frac{p'(t)P(t)}{p^2(t)} = \frac{(4t^3 + 3t^2)(\frac{t^5}{5} + \frac{t^4}{4})}{(t^4 + t^3)^2} = \frac{16t^8 + 32t^7 + 15t^6}{20(t^8 + 2t^7 + t^6)} \geq \frac{3}{4} = c > \frac{1}{2},$$

since

$$16t^8 + 32t^7 + 15t^6 \geq 15t^8 + 30t^7 + 15t^6.$$

Further

$$\frac{xf(x)}{F(x)} = 4 = \frac{2}{2c-1}, \quad x \in [-1,0) \cup (0,1].$$

Hence, for each $u_0 \in [-1,0) \cup (0,1]$ and p, f given by (3.24), (3.25), problem (1.2), (1.11) has a damped Kneser solution u. We can apply asymptotic formulas for u and u' since p satisfies (1.14) and (2.12) with $n = 5$ and f fulfils (1.15) with $r = 3$. These formulas are given by

$$\limsup_{t\to\infty} t|u(t)| < \infty, \quad \limsup_{t\to\infty} t^2|u'(t)| < \infty,$$

according to Theorems 2.1 and 2.3.

In following examples we illustrate other types of function p which appear to satisfy condition for existence of Kneser solutions (3.12)

$$\frac{p'(t)P(t)}{p^2(t)} = c > \frac{1}{2}, \quad t \in (0,\infty).$$

Example 3.9 Let us consider problem (1.2), (1.11), where

$$p(t) = t^3 + t\cos(t), \quad t \in [0,\infty), \tag{3.26}$$

$$f(x) = \begin{cases} -\frac{x+3}{2} & \text{for } x < -1 \\ x^4 \operatorname{sgn} x & \text{for } x \in [-1,1] \\ 2 - x & \text{for } x > 1. \end{cases} \tag{3.27}$$

The graph (Figure 1) of the function

$$\frac{p'(t)P(t)}{p^2(t)} = \frac{3t^2 + \cos(t) - t\sin(t)}{(t^3 + t\cos(t))^2}\left(\frac{t^4}{4} + \cos(t) + t\sin(t)\right)$$

shows that condition (3.12) is satisfied with $c = 0.7$

$$\frac{p'(t)P(t)}{p^2(t)} \geq 0.7, \quad t \in (0,\infty). \tag{3.28}$$

Since

$$\frac{xf(x)}{F(x)} = 5 = \frac{2}{2c-1}, \quad x \in [-1,0) \cup (0,1],$$

problem (1.2), (1.11) has for each $u_0 \in [-1,0) \cup (0,1]$ a damped Kneser solution u, according to Theorems 3.4, 3.5. Moreover, we can apply Theorems 2.1 and 2.3. Function p satisfies conditions (1.14) and (2.12) with $n = 4$ and function f fulfils (1.15) with $r = 4$. Therefore, asymptotic formulas for u and u' are given by

$$\limsup_{t\to\infty} t^{2/3}|u(t)| < \infty, \quad \limsup_{t\to\infty} t^{5/3}|u'(t)| < \infty.$$

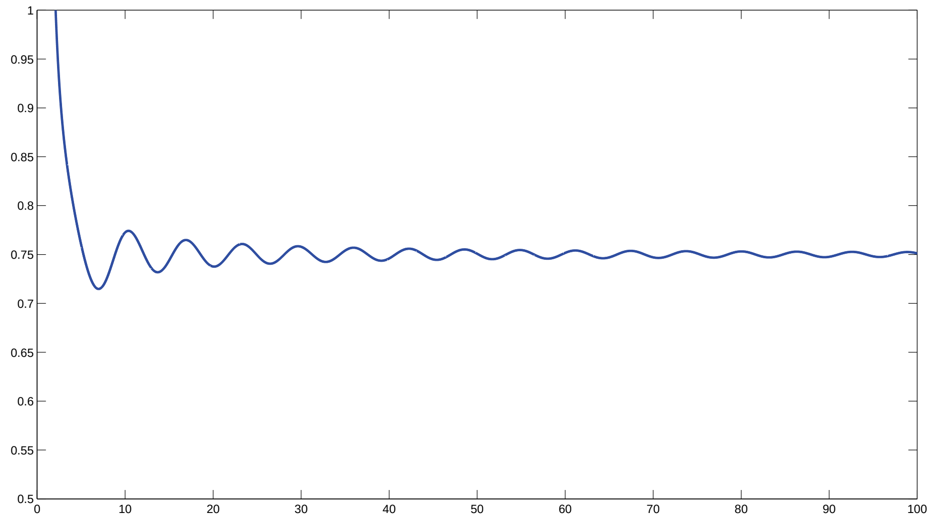

Figure 1: Condition (3.12) for $p(t) = t^3 + t\cos(t)$

Example 3.10 Let us consider problem (1.2), (1.11), where

$$p(t) = \frac{t^3}{1+t}, \quad t \in [0,\infty), \tag{3.29}$$

$$f(x) = \begin{cases} -\frac{x+3}{2} & \text{for } x < -1 \\ x^5 & \text{for } x \in [-1,1] \\ 2-x & \text{for } x > 1. \end{cases} \tag{3.30}$$

The graph (Figure 2) of the function

$$\begin{aligned} \frac{p'(t)P(t)}{p^2(t)} &= \frac{t^2(2t+3)}{(t+1)^2}\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(t+1)\right)\frac{(1+t)^2}{t^6} \\ &= \frac{1}{6t^4}\big(4t^4 + 3t^2 + 18t - 12t\log(t+1) - 18\log(t+1)\big) \end{aligned}$$

shows that

$$\frac{p'(t)P(t)}{p^2(t)} \text{ is monotonous on } (0,\infty) \text{ with } \lim_{t\to\infty}\frac{p'(t)P(t)}{p^2(t)} = \frac{2}{3}. \tag{3.31}$$

We put $c = \frac{2}{3}$, then $\frac{2}{2c-1} = 6$ and condition (3.23) with $r = 5$ is satisfied. Therefore, problem (1.2), (1.11) has for each $u_0 \in [-1,0) \cup (0,1]$ a damped Kneser solution, due to Theorem 3.4 and 3.5. In addition, assumptions (1.14), (2.12) and (1.15) are fulfilled with $n = 3$, $r = 5$. Therefore, the asymptotic

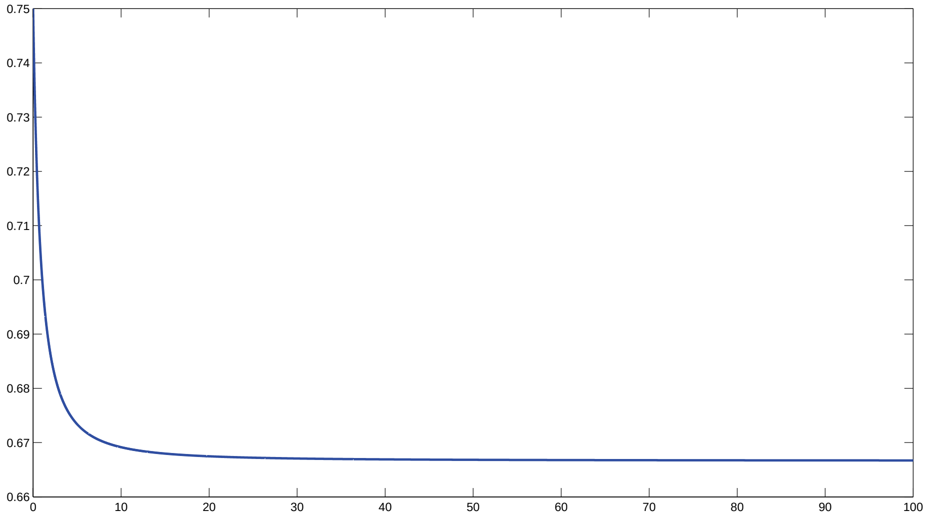

Figure 2: Condition (3.12) for $p(t) = \frac{t^3}{1+t}$

formulas

$$\limsup_{t\to\infty} t^{1/2}|u(t)| < \infty, \quad \limsup_{t\to\infty} t^{3/2}|u'(t)| < \infty.$$

hold, according to Theorems 2.1, 2.3.

Remark 3.11 We are aware that analytical proofs of estimate (3.28) and monotonicity (3.31) cannot be replaced by any graph. However, we are able to illustrate the fulfilment of assumption (3.12) graphically only.

References

[1] Abraham, F. F.: Homogeneous Nucleation Theory. *Acad. Press*, New York, 1974.

[2] Bartušek, M., Cecchi, M., Došlá, Z., Marini, M.: *On oscillatory solutions of quasilinear differential equations.* J. Math. Anal. Appl. **320** (2006), 108–120.

[3] Bongiorno, V., Scriven, L. E., Davis, H. T.: *Molecular theory of fluid interfaces.* J. Colloid and Interface Science **57** (1967), 462–475.

[4] Cecchi, M., Marini, M., Villari, G.: *On some classes of continuable solutions of a nonlinear differential equation.* J. Differential Equations **118** (1995), 403–419.

[5] Cecchi, M., Marini, M., Villari, G.: *Comparison results for oscillation of nonlinear differential equations.* NoDea **6** (1999), 173–190.

[6] Derrick, G. H.: *Comments on nonlinear wave equations as models for elementary particles.* J. Math. Physics **5** (1965), 1252–1254.

[7] Fife, P. C.: *Mathematical Aspects of Reacting and Diffusing Systems.* Lecture notes in Biomathematics *Springer* **28** (1979), 223–224.

[8] Fischer, R. A.: *The wave of advance of advantageous genes.* Journ. of Eugenics **7** (1937), 355–369.

[9] Gouin, H., Rotoli, G.: *An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids.* Mech. Research Communic. **24** (1997), 255–260.

[10] Ho, L. F.: *Asymptotic behavior of radial oscillatory solutions of a quasilinear elliptic equation.* Nonlinear Analysis **41** (2000), 573–589.

[11] Jaroš, J., Kusano, T., Tanigawa, T.: *Nonoscillatory half-linear differential equations and generalized Karamata functions.* Nonlinear Analysis **64** (2006), 762–787.

[12] Kiguradze, I., Chanturia, T.: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. *Kluwer Acad. Publ.*, Dordrecht, 1993.

[13] Kulenović, M. R. S., Ljubović, Ć.: *All solutions of the equilibrium capillary surface equation are oscillatory.* Applied Mathematics Letters **13** (2000), 107–110.

[14] Kusano, T., Manojlović, J. V.: *Asymptotic analysis of Emden-Fowler differential equations in the framework of regular variation.* Annali di Matematica Pura ed Applicata **190** (2011), 619–644.

[15] Kwong, M. K., Wong, J. S. W.: *A nonoscillation theorem for sublinear Emden-Fowler equations.* Nonlinear Analysis **64** (2006), 1641–1646.

[16] Kwong, M. K., Wong, J. S. W.: *A nonoscillation theorem for superlinear Emden–Fowler equations with near-critical coefficients.* J. Differential Equations **238** (2007), 18–42.

[17] Li, W. T.: *Oscillation of certain second-order nonlinear differential equations.* J. Math. Anal. Appl. **217** (1998), 1–14.

[18] Lima, P. M., Chemetov, N. V., Konyukhova, N. B., Sukov, A. I.: *Analytical–numerical investigation of bubble-type solutions of nonlinear singular problems.* J. Comp. Appl. Math. **189** (2006), 260–273.

[19] Linde, A. P.: Particle Physics and Inflationary Cosmology. *Harwood Academic*, Chur, Switzerland, 1990.

[20] Ou, C. H., Wong, J. S. W.: *On existence of oscillatory solutions of second order Emden–Fowler equations.* J. Math. Anal. Appl. **277** (2003), 670–680.

[21] O'Regan, D.: Existence theory for nonlinear ordinary differential equations. *Kluwer*, Dordrecht, 1997.

[22] Rachůnková, I., Rachůnek, L.: *Asymptotic formula for oscillatory solutions of some singular nonlinear differential equation.* Abstract and Applied Analysis **2011** (2011), 1–9.

[23] Rachůnková, I., Tomeček, J.: *Bubble-type solutions of nonlinear singular problem.* Mathematical and Computer Modelling **51** (2010), 658–669.

[24] Rachůnková, I., Rachůnek, L., Tomeček, J.: *Existence of oscillatory solutions of singular nonlinear differential equations.* Abstract and Applied Analysis **2011** (2011), 20 pages.

[25] Rachůnková, I., Tomeček, J.: *Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics.* Nonlinear Analysis **72** (2010), 2114–2118.

[26] Rachůnková, I., Tomeček, J.: *Homoclinic solutions of singular nonautonomous second order differential equations.* Boundary Value Problems **2009** (2009), 1–21.

[27] Rohleder, M.: *On the existence of oscillatory solutions of the second order nonlinear ODE.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **51**, 2 (2012), 107–127.

[28] van der Waals, J. D., Kohnstamm, R.: *Lehrbuch der Thermodynamik.* **1**, Leipzig, 1908.

[29] Wong, J. S. W.: *Second–order nonlinear oscillations: A case history.* In: Proceedings of the Conference on Differential & Difference Equations and Applications *Hindawi* (2006), 1131–1138.

[30] Wong, P. J. Y., Agarwal, R. P.: *Oscillatory behavior of solutions of certain second order nonlinear differential equations.* J. Math. Anal. Appl. **198** (1996), 337–354.

[31] Wong, P. J. Y., Agarwal, R. P.: *The oscillation and asymptotically monotone solutions of second order quasilinear differential equations.* Appl. Math. Comput. **79** (1996), 207–237.