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Fekete–Szegö Problem for a New Class of Analytic Functions Defined by Using a Generalized Differential Operator

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Abstract

In this paper, we obtain Fekete–Szegö inequalities for a generalized class of analytic functions $f(z) \in \mathcal{A}$ for which $1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right)$ $(\alpha, \beta, \lambda, \delta \geq 0;\ \beta > \alpha;\ \lambda > \delta;\ b \in \mathbb{C}^*;\ n \in \mathbb{N}_0;\ z \in U)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis.

Key words: analytic, subordination, Fekete–Szegö problem

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U), \tag{1.1}$$

which are analytic in the open unit disc $U = \{z\colon z \in \mathbb{C} \text{ and } |z| < 1\}$. Further let S denote the family of functions of the form (1.1) which are univalent in U.

A classical theorem of Fekete–Szegö [7] states that, for $f(z) \in S$ given by (1.1) that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases} \tag{1.2}$$

The result is sharp.

Given two functions $f(z)$ and $g(z)$, which are analytic in U with $f(0) = g(0)$, the function $f(z)$ is said to be subordinate to $g(z)$ in U if there exists a function $w(z)$, analytic in U, such that $w(0) = 0$ and $|w(z)| < 1\, (z \in U)$ and $f(z) = g(w(z))$ $(z \in U)$. We denote this subordination by $f(z) \prec g(z)$ in U (see [13]).

Let $\varphi(z)$ be an analytic function with positive real part on U, which satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$, and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*(\varphi)$ be the class of functions $f(z) \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in U), \tag{1.3}$$

and $C(\varphi)$ be the class of functions $f(z) \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \quad (z \in U). \tag{1.4}$$

The classes of $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [12]. The familiar class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ are the special cases of $S^*(\varphi)$ and $C(\varphi)$, respectively, when

$$\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1).$$

Ma and Minda [12] have obtained the Fekete–Szegö problem for the functions in the class $C(\varphi)$. For a function $f(z) \in \mathcal{S}$, Ramadan and Darus [18] introduced the generalized differential operator $D^n_{\alpha,\beta,\lambda,\delta}$ as following:

$$D^0_{\alpha,\beta,\lambda,\delta} f(z) = f(z),$$

$$\begin{aligned} D^1_{\alpha,\beta,\lambda,\delta} f(z) &= [1 - (\lambda - \delta)(\beta - \alpha)]\, f(z) + (\lambda - \delta)(\beta - \alpha)\, zf'(z) \\ &= z + \sum_{k=2}^{\infty} [(\lambda - \delta)(\beta - \alpha)(k - 1) + 1]\, a_k z^k, \end{aligned}$$

$$D^n_{\alpha,\beta,\lambda,\delta} f(z) = D^1_{\alpha,\beta,\lambda,\delta}\left(D^{n-1}_{\alpha,\beta,\lambda,\delta} f(z)\right),$$

$$D^n_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} [(\lambda - \delta)(\beta - \alpha)(k - 1) + 1]^n\, a_k z^k, \tag{1.5}$$

$$(\alpha, \beta, \lambda, \delta \geq 0;\ \delta \geq 0;\ \beta > \alpha;\ \lambda > \delta;\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, 3, \ldots\}).$$

Remark 1 (i) Taking $\alpha = 0$, then operator $D^n_{0,\beta,\lambda,\delta} = D^n_{\beta,\lambda,\delta}$, was introduced and studied by Darus and Ibrahim [6];

(ii) Taking $\alpha = \delta = 0$ and $\beta = 1$, then operator $D^n_{0,1,\lambda,0} = D^n_{\lambda}$, was introduced and studied by Al-Oboudi [1];

(iii) Taking $\alpha = \delta = 0$ and $\lambda = \beta = 1$, then operator $D^n_{0,1,1,0} = D^n$, was introduced and studied by Salagean [20].

Using the generalized operator $D^n_{\alpha,\beta,\lambda,\delta}$ we introduce a new class of analytic functions as following:

Definition 1 For $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, the class $G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$ consists of all functions $f(z) \in \mathcal{A}$ satisfying the following subordination:

$$1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) \prec \varphi(z), \tag{1.6}$$

$$(\alpha, \beta, \lambda, \delta \geq 0;\ \beta > \alpha;\ \lambda > \delta;\ n \in \mathbb{N}_0;\ z \in U).$$

Specializing the parameters α, β, λ, δ, n, b and $\varphi(z)$, we obtain the following subclasses studied by various authors:

(i) $G^{n,1}_{\alpha,\beta,\lambda,\delta}(\varphi) = M^n_{\alpha,\beta,\lambda,\delta}(\varphi)$ (see Ramadan and Darus [18]);

(ii) $G^{n,b}_{0,1,1,0}(\varphi) = H_{n,b}(\varphi)$ (see Aouf and Silverman [4]);

(iii) $G^{0,b}_{0,1,1,0}(\varphi) = S^*_b(\varphi)$ and $G^{1,b}_{0,1,1,0}(\varphi) = C_b(\varphi)$
(see Ravichandran et al. [19]);

(iv) $G^{n,b}_{0,1,1,0}\left(\frac{1+z}{1-z}\right) = S^n(b)$ (see Aouf et al. [2]);

(v) $G^{0,b}_{0,1,1,0}\left(\frac{1+z}{1-z}\right) = S(b)$ (see Nasr and Aouf [17] see also Aouf et al. [3]);

(vi) $G^{1,b}_{0,1,1,0}\left(\frac{1+z}{1-z}\right) = C(b)$ (see Nasr and Aouf [14] see also Aouf et al. [3]);

(vii) $G^{0,(1-\rho)\cos\eta e^{-i\eta}}_{0,1,1,0}\left(\frac{1+z}{1-z}\right) = S^\eta(\rho)$ $\left(|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1\right)$
(see Libera [10] see also Keogh and Merkes [9]);

(viii) $G^{1,(1-\rho)\cos\eta e^{-i\eta}}_{0,1,1,0}\left(\frac{1+z}{1-z}\right) = C^\eta(\rho)$ $\left(|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1\right)$ (see Chichra [5]).

Also we note that for additional choices of parameters we have the following new subclasses of $\mathcal{A}$:

(i)

$$G^{n,b}_{\alpha,\beta,\lambda,\delta}\left(\frac{1+Az}{1+Bz}\right) = S^{n,b}_{\alpha,\beta,\lambda,\delta}(A,B)$$

$$= \left\{ f(z) \in \mathcal{A}\colon 1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}\right.$$

$$\left.(-1 \le B < A \le 1;\ \alpha,\beta,\lambda,\delta \ge 0;\ \beta > \alpha;\ \lambda > \delta;\ n \in \mathbb{N}_0;\ z \in U)\right\};$$

(ii)

$$G^{n,b}_{\alpha,\beta,\lambda,\delta}\left(\frac{1+(1-2\rho)\,z}{1-z}\right) = S^{n,b}_{\alpha,\beta,\lambda,\delta}(\rho)$$

$$= \left\{ f(z) \in \mathcal{A}\colon \operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right)\right\} > \rho\right.$$

$$\left.(\alpha,\beta,\lambda,\delta \ge 0;\ \beta > \alpha;\ \lambda > \delta;\ 0 \le \rho < 1;\ n \in \mathbb{N}_0;\ z \in U)\right\};$$

(iii)

$$G^{n,(1-\rho)\cos\eta e^{-i\eta}}_{\alpha,\beta,\lambda,\delta}(\varphi) = S^{n,\rho,\eta}_{\alpha,\beta,\lambda,\delta}(\varphi)$$

$$= \left\{ f(z) \in \mathcal{A}\colon \frac{e^{i\eta}\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - \rho\cos\eta - i\sin\eta}{(1-\rho)\cos\eta} \prec \varphi(z)\right.$$

$$\left.\left(|\eta| < \tfrac{\pi}{2};\ \alpha,\beta,\lambda,\delta \ge 0;\ \beta > \alpha;\ \lambda > \delta;\ 0 \le \rho < 1;\ n \in \mathbb{N}_0;\ z \in U\right)\right\}.$$

In this paper, we obtain the Fekete–Szegö inequalities for functions in the class $G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$.

2 Fekete–Szegö problem

Unless otherwise mentioned, we assume in the reminder of this paper that $\alpha,\beta,\lambda,\delta \ge 0$, $\beta > \alpha$, $\lambda > \delta$, $b \in \mathbb{C}^*$ and $z \in U$.

To prove our results, we shall need the following lemmas:

Lemma 1 [12] *If* $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ $(z \in U)$ *is a function with positive real part in* U *and* μ *is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \le 2\max\{1; |2\mu - 1|\}. \tag{2.1}$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z)=\frac{1+z}{1-z} \quad (z\in U). \tag{2.2}$$

Lemma 2 [12] *If $p_1(z)=1+c_1z+c_2z^2+\dots$ is a function with positive real part in U, then*

$$\left|c_2-\nu c_1^2\right| \le \begin{cases} -4\nu+2, & \text{if } \nu\le 0,\\ 2, & \text{if } 0\le\nu\le 1,\\ 4\nu-2, & \text{if } \nu\ge 1.\end{cases}$$

When $\nu<0$ or $\nu>1$, the equality holds if and only if

$$p_1(z)=\frac{1+z}{1-z}$$

or one of its rotations. If $0<\nu<1$, then the equality holds if and only if

$$p_1(z)=\frac{1+z^2}{1-z^2}$$

or one of its rotations. If $\nu=0$, the equality holds if and only if

$$p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0\le\gamma\le 1),$$

or one of its rotations. If $\nu=1$, the equality holds if and only if

$$\frac{1}{p_1(z)}=\left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0\le\gamma\le 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0<\nu<1$:

$$\left|c_2-\nu c_1^2\right|+\nu\left|c_1\right|^2\le 2 \quad (0<\nu<\tfrac{1}{2}),$$

and

$$\left|c_2-\nu c_1^2\right|+(1-\nu)\left|c_1\right|^2\le 2 \quad (\tfrac{1}{2}<\nu<1).$$

Using Lemma 1, we have the following theorem:

Theorem 1 *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\dots$, where $\varphi(z)\in\mathcal{A}$ and $\varphi'(0)>0$. If $f(z)$ given by (1.1) belongs to the class $G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$ and if μ is a complex number, then*

$$\left|a_3-\mu a_2^2\right|\le\frac{|b|B_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}$$
$$\times\max\left\{1,\left|\frac{B_2}{B_1}+\left(1-\frac{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2n}}\mu\right)bB_1\right|\right\}. \tag{2.3}$$

The result is sharp.

Proof If $f(z) \in G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$, then there exists a Schwarz function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) = \varphi(w(z)). \tag{2.4}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \dots \tag{2.5}$$

Since $w(z)$ is a Schwarz function, we see that $\operatorname{Re}\{p_1(z)\} > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by:

$$p(z) = 1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) = 1 + b_1 z + b_2 z^2 + \dots \tag{2.6}$$

In view of the equations (2.4), (2.5) and (2.6), we have

$$\begin{aligned} p(z) &= \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = \varphi\left(\frac{c_1 z + c_2 z^2 + \dots}{2 + c_1 z + c_2 z^2 + \dots}\right) \\ &= \varphi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \dots\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right] z^2 + \dots \end{aligned} \tag{2.7}$$

Thus

$$b_1 = \frac{1}{2}B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2. \tag{2.8}$$

Since

$$1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) = 1 + \left\{\frac{1}{b}\left(\left[(\lambda - \delta)(\beta - \alpha) + 1\right]^n a_2\right)\right\} z$$
$$+ \left\{\frac{1}{b}\left(2\left[2(\lambda - \delta)(\beta - \alpha) + 1\right]^n a_3 - \left[(\lambda - \delta)(\beta - \alpha) + 1\right]^{2n} a_2^2\right)\right\} z^2 + \dots$$

Then from (2.6) and (2.8), we obtain

$$a_2 = \frac{bB_1 c_1}{2\left[(\lambda - \delta)(\beta - \alpha) + 1\right]^n}, \tag{2.9}$$

and

$$a_3 = \frac{bB_1 c_2}{4\left[2(\lambda - \delta)(\beta - \alpha) + 1\right]^n} + \frac{c_1^2}{8\left[2(\lambda - \delta)(\beta - \alpha) + 1\right]^n}\left[b^2 B_1^2 - b(B_1 - B_2)\right]. \tag{2.10}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}\left[c_2 - \nu c_1^2\right], \tag{2.11}$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}}\mu - 1\right)bB_1\right]. \tag{2.12}$$

Our result now follows by an application of Lemma 1. The result is sharp for the function $f(z)$ given by

$$1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) = \varphi(z^2), \tag{2.13}$$

or

$$1 + \frac{1}{b}\left(\frac{z\left(D^n_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) = \varphi(z). \tag{2.14}$$

This completes the proof of Theorem 1. □

Remark 2 (i) Taking $n = 0$ in Theorem 1, we improve the result obtained by Ravichandran et al. [19, Theorem 4.1];

(ii) Taking $\alpha = \delta = 0$, $\beta = \lambda = 1$, $b = (1-\rho)\cos\eta e^{-i\eta}$ ($|\eta| < \frac{\pi}{2}$, $0 \le \rho < 1$) and $\varphi(z) = \frac{1+z}{1-z}$ (equivalently $B_1 = B_2 = 2$) in Theorem 1, we obtain the result obtained by Goyal and Kumar [8, Corollary 2.10];

(iii) Taking $b = (1-\rho)\cos\eta e^{-i\eta}$ ($|\eta| < \frac{\pi}{2}$, $0 \le \rho < 1$), $n = 0$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result obtained by Keogh and Merkes [9, Thm 1];

(iv) Taking $\alpha = \delta = 0$ and $\beta = \lambda = 1$ in Theorem 1, we obtain the result obtained by Aouf and Silverman [4, Theorem 1].

Also by specializing the parameters in Theorem 1, we obtain the following new sharp results.

Putting $b = 1$ in Theorem 1, we obtain the following corollary:

Corollary 1 *If $f(z)$ given by* (1.1) *belongs to the class $M^n_{\alpha,\beta,\lambda,\delta}(\varphi)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}$$
$$\times \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}}\mu\right)B_1\right|\right\}. \tag{2.15}$$

The result is sharp.

Putting $\varphi(z) = \frac{1+Az}{1-Bz}$ $(-1 \leq B < A \leq 1)$ (or equivalently, $B_1 = A - B$ and $B_2 = -B(A-B)$) in Theorem 1, we obtain the following corollary:

Corollary 2 *If $f(z)$ given by (1.1) belongs to the class $S^{n,b}_{\alpha,\beta,\lambda,\delta}(A,B)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)\,|b|}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}$$

$$\times \max\left\{1, \left|\left(1 - \frac{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2n}}\mu\right)(A-B)\,b - B\right|\right\}. \tag{2.16}$$

The result is sharp.

Putting $\varphi(z) = \frac{1+(1-2\rho)z}{1-z}$ $(0 \leq \rho < 1)$ in Theorem 1, we obtain the following corollary:

Corollary 3 *If $f(z)$ given by (1.1) belongs to the class $S^{n,b}_{\alpha,\beta,\lambda,\delta}(\rho)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(1-\rho)\,|b|}{\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}$$

$$\times \max\left\{1, \left|2\left(1 - \frac{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2n}}\mu\right)(1-\rho)b + 1\right|\right\}. \tag{2.17}$$

The result is sharp.

Putting $b = (1-\rho)\cos\eta e^{-i\eta}$ $(|\eta| < \frac{\pi}{2},\ 0 \leq \rho < 1)$ in Theorem 1, we obtain the following corollary:

Corollary 4 *If $f(z)$ given by (1.1) belongs to the class $S^{n,\rho,\eta}_{\alpha,\beta,\lambda,\delta}(\varphi)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1\left(1-\rho\right)\cos\eta}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}$$

$$\times \max\left\{1, \left|\frac{B_2}{B_1}e^{i\eta} + \left(1 - \frac{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^{2n}}\mu\right)(1-\rho)B_1\cos\eta\right|\right\}. \tag{2.18}$$

The result is sharp.

Putting $\alpha = \delta = 0$, $\beta = \lambda = 1$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of Aouf et al. [2, Theorem 3, with $m = 1$]:

Corollary 5 *If $f(z)$ given by (1.1) belongs to the class $S^n(b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|}{3^n}\max\left\{1, \left|1 + 2\left(1 - 2\left(\tfrac{3}{4}\right)^n\mu\right)b\right|\right\}. \tag{2.19}$$

The result is sharp.

Putting $n = 0$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of and Nasr and Aouf [17, Theorem 2] see also Nasr and Aouf [16, Theorem 1, with $m = 1$]:

Corollary 6 *If $f(z)$ given by (1.1) belongs to the class $S(b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le |b| \max\left\{1, |1 + 2(1-2\mu)b|\right\}. \tag{2.20}$$

The result is sharp.

Putting $\alpha = \delta = 0$, $\beta = \lambda = 1$, $n = 1$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of Nasr and Aouf [15, Theorem 1, with $m = 1$] see also Nasr and Aouf [14]:

Corollary 7 *If $f(z)$ given by (1.1) belongs to the class $C(b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3} |b| \max\left\{1, \left|1 + 2\left(1 - \tfrac{3}{2}\mu\right) b\right|\right\}. \tag{2.21}$$

The result is sharp.

Putting $\alpha = \delta = 0$, $\beta = \lambda = 1$, $n = 0$, $b = (1-\rho)\cos\eta e^{-i\eta}$ ($|\eta| < \frac{\pi}{2}$, $0 \le \rho < 1$) and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of Keogh and Merkes [9, Theorem 1]:

Corollary 8 *If $f(z)$ given by (1.1) belongs to the class $S^\eta(\rho)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le (1-\rho)\cos\eta \max\left\{1, \left|2(2\mu - 1)(1-\rho)\cos\eta - e^{i\eta}\right|\right\}. \tag{2.22}$$

The result is sharp.

Putting $\alpha = \delta = 0$, $\beta = \lambda = 1$, $n = 1$, $b = (1-\rho)\cos\eta e^{-i\eta}$ ($|\eta| < \frac{\pi}{2}$, $0 \le \rho < 1$) and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of Libera and M. Ziegler [11, Lemma 1, with $\rho = 0$] see also Chichra [5]:

Corollary 9 *If $f(z)$ given by (1.1) belongs to the class $C^\eta(\rho)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3}(1-\rho)\cos\eta \max\left\{1, \left|2\left(\tfrac{3}{2}\mu - 1\right)(1-\rho)\cos\eta - e^{i\eta}\right|\right\}. \tag{2.23}$$

The result is sharp.

Using Lemma 2, we have the following theorem:

Theorem 2 *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ $(b > 0;\ B_i > 0;\ i \in \mathbb{N})$. *Also let*

$$\sigma_1 = \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}\left(B_2 - B_1 + bB_1^2\right)}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n bB_1^2},$$

and

$$\sigma_2 = \frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}\left(B_2 + B_1 + bB_1^2\right)}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n bB_1^2}.$$

If $f(z)$ *is given by* (1.1) *belongs to the class* $G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$, *then we have the following sharp results:*

(i) If $\mu \le \sigma_1$, *then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n} \times \left\{B_2 - \left(2\frac{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}}\mu - 1\right) bB_1^2\right\}. \quad (2.24)$$

(ii) If $\sigma_1 \le \mu \le \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{bB_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}. \quad (2.25)$$

(iii) If $\mu \ge \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n} \times \left\{-B_2 + \left(2\frac{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}}\mu - 1\right) bB_1^2\right\}. \quad (2.26)$$

Proof For $f(z) \in G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$, $p(z)$ given by (2.6) and $p_1(z)$ given by (2.5), then a_2 and a_3 are given as same as in Theorem 1. Also

$$a_3 - \mu a_2^2 = \frac{bB_1}{4\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}\left[c_2 - \nu c_1^2\right], \quad (2.27)$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}}\mu - 1\right) bB_1\right]. \quad (2.28)$$

First, if $\mu \le \sigma_1$, then we have $\nu \le 0$, then by applying Lemma 2 to equality (2.27), we have

$$\left|a_3 - \mu a_2^2\right| \le$$
$$\le \frac{b}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}\left\{B_2 - \left(2\frac{\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}}\mu - 1\right) bB_1^2\right\},$$

which is evidently inequality (2.24) of Theorem 2.

If $\mu = \sigma_1$, then we have $\nu = 0$, therefore equality holds if and only if

$$p_1(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z} \quad (0 \le \gamma \le 1;\ z \in U).$$

Next, if $\sigma_1 \le \mu \le \sigma_2$, we note that

$$\max\left\{\frac{1}{2}\left[1 - \frac{B_2}{B_1} + \left(\frac{2\,[2(\lambda-\delta)(\beta-\alpha)+1]^n}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}}\mu - 1\right) bB_1\right]\right\} \le 1, \tag{2.29}$$

then applying Lemma 2 to equality (2.27), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{bB_1}{2\,[2(\lambda-\delta)(\beta-\alpha)+1]^n},$$

which is evidently inequality (2.25) of Theorem 2.

If $\sigma_1 < \mu < \sigma_2$, then we have

$$p_1(z) = \frac{1+z^2}{1-z^2}.$$

Finally, If $\mu \ge \sigma_2$, then we have $\nu \ge 1$, therefore by applying Lemma 2 to (2.27), we have

$$\left|a_3 - \mu a_2^2\right| \le$$

$$\le \frac{b}{2\,[2(\lambda-\delta)(\beta-\alpha)+1]^n}\left\{-B_2 + \left(2\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^n}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}}\mu - 1\right) bB_1^2\right\},$$

which is evidently inequality (2.26) of Theorem 2.

If $\mu = \sigma_2$, then we have $\nu = 1$, therefore equality holds if and only if

$$\frac{1}{p_1(z)} = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z} \quad (0 \le \gamma \le 1;\ z \in U).$$

To show that the bounds are sharp, we define the functions $K_\varphi^s (s \ge 2)$ by

$$1 + \frac{1}{b}\left(\frac{z(D^n_{\alpha,\beta,\lambda,\delta}K_\varphi^s(z))'}{D^n_{\alpha,\beta,\lambda,\delta}K_\varphi^s(z)} - 1\right) = \varphi\big(z^{s-1}\big), \quad K_\varphi^s(0) = 0 = K_\varphi'^s(0) - 1, \tag{2.30}$$

and the functions F_t and G_t $(0 \le t \le 1)$ by

$$1 + \frac{1}{b}\left(\frac{z(D^n_{\alpha,\beta,\lambda,\delta}F_t(z))'}{D^n_{\alpha,\beta,\lambda,\delta}F_t(z)} - 1\right) = \varphi\left(\frac{z(z+t)}{1+tz}\right), \quad F_t(0) = 0 = F_t'(0) - 1, \tag{2.31}$$

and

$$1+\frac{1}{b}\left(\frac{z(D^{n}_{\alpha,\beta,\lambda,\delta}G_t(z))'}{D^{n}_{\alpha,\beta,\lambda,\delta}G_t(z)}-1\right)=\varphi\left(-\frac{z(z+t)}{1+tz}\right),\quad G_t(0)=0=G_t'(0)-1. \tag{2.32}$$

Cleary the functions K^s_φ, F_t and $G_t \in G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$. Also we write $K_\varphi = K^2_\varphi$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is K^3_φ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_t or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_t or one of its rotations. □

Remark 3 (i) Taking $b = 1$ in Theorem 2, we improve the result obtained by Ramadan and Darus [18, Theorem 1];

(ii) Taking $\alpha = \delta = 0$ and $\beta = \lambda = 1$ in Theorem 2, we obtain the result obtained by Goyal and Kumar [8, Corollary 2.7] and Aouf and Silverman [4, Theorem 2].

Also, using Lemma 2 we have the following theorem:

Theorem 3 *For $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$ $(b > 0;\ B_i > 0;\ i \in \mathbb{N})$ and $f(z)$ given by (1.1) belongs to the class $G^{n,b}_{\alpha,\beta,\lambda,\delta}(\varphi)$ and $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 2, Theorem 2 can be improved. Let*

$$\sigma_3=\frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}\left(B_2+bB_1^2\right)}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{n}bB_1^2},$$

(i) If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3-\mu a_2^2\right|+\frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{n}bB_1}$$
$$\times\left\{1-\frac{B_2}{B_1}+\left(2\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^{n}}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}}-1\right)bB_1\right\}|a_2|^2$$
$$\le\frac{bB_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{n}}; \tag{2.33}$$

(ii) If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3-\mu a_2^2\right|+\frac{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{n}bB_1}$$
$$\times\left\{1+\frac{B_2}{B_1}-\left(2\frac{[2(\lambda-\delta)(\beta-\alpha)+1]^{n}}{[(\lambda-\delta)(\beta-\alpha)+1]^{2n}}\mu-1\right)bB_1\right\}|a_2|^2$$
$$\le\frac{bB_1}{2\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^{n}}. \tag{2.34}$$

Proof For the values of $\sigma_1 \leq \mu \leq \sigma_3$, we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)\left|a_2\right|^2 = \frac{bB_1}{4\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}\left|c_2 - \nu c_1^2\right|$$

$$+\left(\mu - \frac{\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{2n}\left(B_2 - B_1 + bB_1^2\right)}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n bB_1^2}\right)\frac{b^2B_1^2}{4\left[(\lambda-\delta)\left(\beta-\alpha\right)+1\right]^{2n}}\left|c_1\right|^2$$

$$= \frac{bB_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2\right)\right\}. \qquad (2.35)$$

Now apply Lemma 2 to equality (2.35), then we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)\left|a_2\right|^2 \leq \frac{bB_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n},$$

which is evidently inequality (2.33) of Theorem 3.

Next, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we have

$$\left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)\left|a_2\right|^2 = \frac{bB_1}{4\left[2(\lambda-\delta)(\beta-\alpha)+1\right]^n}\left|c_2 - \nu c_1^2\right|$$

$$+\left(\frac{\left[(\lambda-\delta)\left(\beta-\alpha\right)+1\right]^{2n}\left(B_2 + B_1 + bB_1^2\right)}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n bB_1^2} - \mu\right)\frac{b^2B_1^2}{4\left[(\lambda-\delta)\left(\beta-\alpha\right)+1\right]^{2n}}\left|c_1\right|^2$$

$$= \frac{bB_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2\right)\right\}. \qquad (2.36)$$

Now apply Lemma 2 to equality (2.36), then we have

$$\left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)\left|a_2\right|^2 \leq \frac{bB_1}{2\left[2\left(\lambda-\delta\right)\left(\beta-\alpha\right)+1\right]^n},$$

which is evidently inequality (2.34). This completes the proof of Theorem 3. □

Remark 4 (i) taking $\alpha = \delta = 0$ and $\beta = \lambda = 1$ in Theorem 3, we improve the result obtained by Goyal and Kumar [8, Remark 2.8];

(ii) taking $b = 1$ in Theorem 3, we improve the result obtained by Ramadan and Darus [18, Remark 2].

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