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Hammerstein–Nemytskii Type Nonlinear Integral Equations on Half-line in Space $L_1(0,+\infty) \cap L_\infty(0,+\infty)$

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Abstract

The paper studies a construction of nontrivial solution for a class of Hammerstein–Nemytskii type nonlinear integral equations on half-line with noncompact Hammerstein integral operator, which belongs to space $L_1(0,+\infty) \cap L_\infty(0,+\infty)$. This class of equations is the natural generalization of Wiener-Hopf type conservative integral equations. Examples are given to illustrate the results. For one type of considering equations continuity and uniqueness of the solution is established.

Wiener–Hopf operator, Hammerstein–Nemytskii equation, Caratheodory condition, one-parameter family of positive solutions, iteration, monotonic increasing and bounded solution

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1 Introduction

In the paper we discuss the questions of solvability and uniqueness of the following class of Hammerstein–Nemytskii type nonlinear integral equation

$$\varphi(x) = A(x,\varphi(x)) + \int_0^\infty K(x-t)B(t,\varphi(t))\,dt, \quad x \geq 0 \tag{1.1}$$

with respect to an unknown real measurable function. Here $A(x,\tau)$ and $B(t,\tau)$ are given measurable real functions, defined on set $\mathbb{R}^+ \times \mathbb{R}$ ($\mathbb{R}^+ \equiv [0,+\infty)$, $\mathbb{R} \equiv (-\infty,+\infty)$) and satisfying some conditions (see Theorem 3.1 and 3.2).

The kernel $K(\tau)$ is a measurable function on $\mathbb{R}$, such that

$$K(\tau) \geq 0,\ \ \tau \in \mathbb{R}, \qquad \int_{-\infty}^{+\infty} K(\tau)\, d\tau = 1, \tag{1.2}$$

$$K \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \qquad \int_{-\infty}^{+\infty} \tau^2 K(\tau)\, d\tau < +\infty, \tag{1.3}$$

$$\nu(K) \equiv \int_{-\infty}^{+\infty} \tau K(\tau)\, d\tau < 0. \tag{1.4}$$

Note that equation (1.1) is the natural generalization of Wiener–Hopf conservative equation ((1.2)–(1.3)) in case where $A(x,\tau) \equiv g(x)$; $B(t,\tau) \equiv \tau$.

It should be noted, that recently in work [9] has been investigated the equation (1.1) in case when $B(t,\tau) = \tau$ and $A(x,\tau) = y(x) - F(x,\tau)$, $y \in L_1(\mathbb{R}^+)$, where the functions $F(x,\tau)$ and $K(z)$ satisfy certain conditions. In work [9] the existence of solution in space $L_2(0,+\infty)$ is proved. Note also that in case when $A(x,\tau) \equiv 0$, $B(t,z) = z^\alpha$ $(0 < \alpha < 1)$ the questions of existence and uniqueness of equation (1.1) has been studied in work [3].

Below we'll investigate equation (1.1) under essentially another assumptions on functions A and B (see conditions of Theorem 3.1 and Theorem 3.2). Especially first we'll not assume compactness of corresponding Hammerstein–Nemytskii type operator and second we'll also consider the complicated case when $A(x,0) = 0$, $B(x,0) = 0$ (see Remark 1).

Using some results of linear theory of Wiener–Hopf integral equations, by means of a'priori estimations and special iteration processes, the existence of positive nontrivial solution in space $L_1(0,+\infty) \cap L_\infty(0,+\infty)$ is proved. In particular case the continuity and uniqueness of solution in certain class of functions are established.

2 Preliminaries

2.1 Factorization of Wiener–Hopf conservative integral operator

Let E be one of the following banach spaces $L_p(0,+\infty)$, $1 \leq p < +\infty$, $L_\infty(0,+\infty)$, $C_M[0,+\infty) \equiv C[0,+\infty) \cap L_\infty(0,+\infty)$, $C_0[0,+\infty)$, where $C_0[0,+\infty)$ is the space of continuous functions on $[0,+\infty)$ with zero limit at infinity.

We consider Wiener–Hopf integral operator

$$(\mathcal{K}f)(x) = \int_0^\infty K(x-t) f(t)\, dt, \quad f \in E \tag{2.1}$$

where kernel function $K(x)$ satisfies conservative condition (1.2).

As it is known (cf. [2]) the operator $\mathcal{K}$ preserves the space E, i.e. it maps every E into itself, moreover in each space from E the inequality holds

$$\|\mathcal{K}\|_E \le \int_{-\infty}^{+\infty} |K(x)|\,dx. \tag{2.2}$$

It follows from results of [1] that operator $I-\mathcal{K}$ (I is the unit operator), under the conditions (1.2) and (1.4) permits the following Volteryan factorization

$$I-\mathcal{K} = (I-V_-)(I-V_+) \tag{2.3}$$

where

$$\begin{aligned}(V_-f)(x) &= \int_x^\infty v_-(t-x)f(t)\,dt\\ (V_+f)(x) &= \int_0^x v_+(x-t)f(t)\,dt\end{aligned} \tag{2.4}$$

$$0 \le v_\pm \in L_1; \quad \gamma_\pm \equiv \int_0^\infty v_\pm(x)\,dx, \quad \gamma_- = 1,\ \gamma_+ < 1. \tag{2.5}$$

Besides we assume that the following conditions hold:

$$\int_{-\infty}^{+\infty} |x|^j K(x)\,dx < +\infty, \quad j = 1,2 \tag{2.6}$$

$$m_1(v_\pm) \equiv \int_0^\infty x v_\pm(x)\,dx < +\infty. \tag{2.7}$$

One of the authors in his recent paper [5] has proved as an auxiliary result that if

$$K \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \quad \text{then } v_\pm \in L_1(0,+\infty) \cap L_\infty(0,+\infty). \tag{2.8}$$

2.2 Wiener–Hopf homogeneous conservative equation

In this subsection we consider the Wiener–Hopf homogeneous equation

$$S(x) = \int_0^\infty K(x-t)S(t)\,dt, \quad S(0)=1,\ x \ge 0. \tag{2.9}$$

As it is known (cf. [5]) the equation (2.9) under conditions (1.2) and (1.4) has a positive, monotonously increasing and bounded solution $S(x)$, such that

$$1 = S(0) \le S(x) \le (1-\gamma_+)^{-1} \tag{2.10}$$

$$S(x) \uparrow (1-\gamma_+)^{-1} \quad \text{as } x \to +\infty. \tag{2.11}$$

Below we essentially use this interesting result.

2.3 Wiener–Hopf–Hammerstein integral equation in conservative case

We consider the following nonlinear integral equation on half-line

$$Q(x) = \int_0^\infty K(x-t)(Q(t) - \omega(t, Q(t)))\, dt, \quad x > 0 \tag{2.12}$$

with respect to an unknown real function $Q(x)$.

Here kernel $K(x)$ satisfies conditions (1.2)–(1.4). The function $\omega(t,z)$, $(t,z) \in \mathbb{R}^+ \times \mathbb{R}$, specifing nonlinearity of equation (2.12) possesses the following properties

1) $\omega \in \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and there exists a number $A_0 > 0$ such that $\omega(t,z)\downarrow$ in z on $[A_0, +\infty)$ for each fixed $t \in \mathbb{R}^+$,
2) $\omega(t,z) \geq 0$, $(t,z) \in \mathbb{R}^+ \times [A_0, +\infty) \equiv \Omega_{A_0}$ and ω satisfies Caratheodory condition on the set Ω_{A_0} in z,
3) there exists a measurable function $\overset{\circ}{\omega} \in L_1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$, $\overset{\circ}{\omega}(z) \geq 0$, $z \in [A_0, +\infty)$, $\overset{\circ}{\omega}\downarrow$ in z on $[A_0, +\infty)$, $m_1(\overset{\circ}{\omega}) \equiv \int_0^\infty x\, \overset{\circ}{\omega}(x) dx < +\infty$, such that

$$\omega(t,z) \leq \overset{\circ}{\omega}(t+z), \quad (t,z) \in \Omega_{A_0}. \tag{2.13}$$

In [5] it is proved that equation (2.12) possesses one parameter family of positive and bounded solution $Q_\gamma(x)$:

$$Q_\gamma(x) > 0,\ x \in \mathbb{R}^+,\ Q_\gamma(x)\uparrow \text{ by } \gamma, \quad Q_\gamma \in L_\infty[0, +\infty), \tag{2.14}$$

$$\lim_{x\to+\infty} Q_\gamma(x) = 2\gamma(1-\gamma_+)^{-1} \tag{2.15}$$

where $\gamma \in \Delta \equiv [\max(\delta, \gamma_0), +\infty)$.

Moreover in [5] it is proved that if $\omega(t,z)\downarrow$ in t on $\mathbb{R}^+$, then $Q_\gamma(x)\uparrow$ in x on $\mathbb{R}^+$.

Here

$$\delta = \operatorname*{esssup}_{x\in\mathbb{R}^+} \rho(x) \tag{2.16}$$

where $\rho(x)$ is a unique positive, bounded (in class of summerable functions) solution of the Wiener–Hopf nonhomogeneous integral equation

$$\rho(x) = 2\,\overset{\circ}{\omega}(x + A_0) + \int_0^\infty K(x-t)\rho(t)\, dt, \quad x > 0, \tag{2.17}$$

$$\lim_{x\to+\infty} \rho(x) = 0, \tag{2.18}$$

By $\gamma_0 \in [A_0, +\infty)$ we denoted a fixed number, for which $\overset{\circ}{\omega}(\gamma_0) < \gamma_0$. Existence of such number immediately follows from the properties of function $\overset{\circ}{\omega}$.

In [5], [6] the following double inequalities for $Q_\gamma(x)$ are also proved:

$$2S_\gamma(x) - \psi_\gamma(x) \leq Q_\gamma(x) \leq 2S_\gamma(x), \quad x > 0, \tag{2.19}$$

where

$$S_\gamma(x) = \gamma S(x); \quad \gamma \in \Delta \tag{2.20}$$

and $\psi_\gamma(x) \geq 0$ is a solution of the following convolution type integral equation

$$\psi_\gamma(x) = 2\,\overset{\circ}{\omega}(x + S_\gamma(x)) + \lambda_\gamma(x) \int_0^\infty K(x-t)\psi_\gamma(t)\,dt, \quad x > 0 \tag{2.21}$$

$$\lambda_\gamma(x) = 1 - \frac{\overset{\circ}{\omega}(x + S_\gamma(x))}{S_\gamma(x)}. \tag{2.22}$$

It follows from properties of S_γ, λ_γ and $\overset{\circ}{\omega}$ that

$$0 \leq \psi_\gamma(x) \leq \rho(x), \quad \gamma \in \Delta, \quad x > 0. \tag{2.23}$$

Therefore

$$\lim_{x\to+\infty} \psi_\gamma(x) = 0, \quad \psi_\gamma \in L_1(0,+\infty) \cap L_\infty(0,+\infty). \tag{2.24}$$

3 Basic nonlinear integral equation

3.1 Formulation of basic theorems

The following theorems are true.

Theorem 3.1 *Let $A(x,\tau)$ and $B(t,z) \equiv B_0(z)$ be given measurable real functions, defined on sets $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}$ respectively. We assume that there exist numbers $\eta > 0$ and $\eta_0 \in (0,\eta)$ such that*

i_1) *for each $x \in \mathbb{R}^+$ the function $A(x,\tau)\uparrow$ in τ on $[\Phi_{\eta_0}(x), \eta] \equiv \Theta(x)$, where*

$$\Phi_\varepsilon(x) \equiv \varepsilon \int_x^\infty K(\tau)d\tau, \quad \varepsilon > 0, \quad x \in \mathbb{R}^+, \tag{3.1}$$

i_2) *$A(x,\tau)$ satisfies Caratheodory condition on set $\mathbb{R}^+ \times [0,\eta]$ by $\tau \in [0,\eta]$*

$$\Phi_{\eta_0}(x) \leq A(x,\tau) \leq \Phi_\eta(x), \quad x \in \mathbb{R}^+, \quad \tau \in \Theta(x) \tag{3.2}$$

i_3) *$B_0(z)\uparrow$ in z on $[0,\eta]$, $B_0 \in C[0,\eta]$ and*

$$0 \leq B_0(z) \leq z, \quad z \in [0,\eta] \tag{3.3}$$

Then under the conditions (1.2)–(1.4) *equation* (1.1) *possesses positive solution, which belongs to $L_1(0,+\infty) \cap L_\infty(0,+\infty)$.*

Theorem 3.2 *Let $A(x,\tau)$ and $B(t,z)$ be given measurable real functions defined on $\mathbb{R}^+ \times \mathbb{R}$. There exist numbers*

$$\eta \geq 2(1-\gamma_0)^{-1} \max(\delta,\gamma_0) \quad \textit{and} \quad \eta_0 \in [0,\eta] \tag{3.4}$$

such that $A(x,\tau)$ satisfies conditions i_1), i_2) from Theorem 3.1. We also assume that

j_1) $B(t,z)\uparrow$ in z on $[0,\eta]$ for each fixed $t \in \mathbb{R}^+$ and satisfies Carateodory condition on set $\mathbb{R}^+ \times [0,\eta]$ by argument $z \in [0,\eta]$,

j_2) the condition

$$0 \le B(t,z) \le z + \omega(t,\eta), \quad (t,z) \in \mathbb{R}^+ \times [0,\eta] \tag{3.5}$$

is fulfilled, where $\omega(t,z)$-satisfies conditions 1)–3) from subsection 2.3.

Then under the equation conditions (1.2)–(1.4), (1.1) *possesses positive solution in $L_1(0,+\infty) \cap L_\infty(0,+\infty)$ space.*

Definition 3.1 (see [5]) Let E be a real banach space, $C \subset E$ be regular cone with positive elements, and let Λ be some nonlinear integral operator, acting in cone C, for which $\Lambda\theta = \theta$, where θ is the zero element of cone. The operator Λ is called critical, if equation $\Lambda x = x$ besides trivial solution has even if one positive solution.

Remark 3.1 If $A(x,0) \equiv 0$ and $B(t,0) \equiv 0$ it follows from Theorems 3.1 and 3.2 that corresponding nonlinear operator

$$(\Lambda f)(x) = A(x, f(x)) + \int_0^\infty K(x-t) B(t, f(t))\, dt$$

is critical.

3.2 Examples of functions A, B_0 and B

First we list some examples of functions $B_0(z)$ and $B(t,z)$.

Example 3.1

$$B_0(z) = \frac{z^p}{\eta^{p-1}}, \quad z \in \mathbb{R},\ \eta > 0,\ p > 1. \tag{3.6}$$

Example 3.2

$$B_0(z) = \frac{2\eta}{\pi} \sin \frac{\pi z}{2\eta}, \quad z \in \mathbb{R},\ \eta > 0. \tag{3.7}$$

Example 3.3

$$B(t,z) = B_0(z) + \omega(t,\eta) q(z)$$

where $q(z)$ is a real function defined on $\mathbb{R}$, moreover $q(0) = 0$, $\eta \in C[0,\eta]$, $q\uparrow$ in z on $[0,\eta]$,

$$0 \le q(z) \le 1 \quad \text{at } z \in [0,\eta],\ \eta \ge 2(1-\gamma_+)^{-1} \max(\delta, \gamma_0).$$

Below we list some examples of $A(x,\tau)$ and discuss one of them in detail.

Example 3.4

$$\begin{gathered} A(x,\tau) \equiv A_1(x,\tau) = \Phi_{\eta_0+\eta_1}(x) \frac{\tau}{\tau + \Phi_{\eta_1}(x)}, \\ (x,t) \in \mathbb{R}^+ \times \mathbb{R},\ \eta_0, \eta_1 > 0,\ \eta \ge \eta_0 + \eta_1. \end{gathered} \tag{3.8}$$

Example 3.5

$$A(x,\tau) \equiv A_2(x,\tau) = \Phi_{\eta_0+\eta_1}(x)\frac{\alpha\tau^2}{(\tau+\Phi_{\eta_1}(x))^2},$$
$$\alpha \geq 1+\frac{\eta_1}{\eta_0},\ \eta \geq \alpha(\eta_0+\eta_1),\ \eta_0,\eta_1>0,\ (x,t)\in\mathbb{R}^+\times\mathbb{R}.$$

Example 3.6

$$A(x,\tau) \equiv A_3(x,\tau) = \frac{A_1(x,\tau)(A_1(x,\tau)+3)}{(A_1(x,\tau)+2)}\frac{(A_2(x,\tau)+2)}{(A_2(x,\tau)+3)}.$$

Example 3.7

$$A(x,\tau) \equiv A_4(x,\tau) = A_1(x,\tau)\left(\ln\frac{A_2(x,\tau)}{A_1(x,\tau)}+1\right).$$

Example 3.8

$$A(x,\tau) \equiv A_5(x,\tau) = \frac{2A_1(x,\tau)A_2(x,\tau)}{A_1(x,\tau)+A_2(x,\tau)}.$$

Below we show that Example 3.4 satisfies all conditions of Theorems 3.1 and 3.2.

We have

$$\frac{\partial A}{\partial\tau} = \Phi_{\eta_0+\eta_1}(x)\frac{\Phi_{\eta_1}(x)}{(\tau+\Phi_{\eta_1}(x))^2} > 0, \quad x\in\mathbb{R}^+,\ \tau\in\Theta(x).$$

Therefore $A\uparrow$ in τ on $\Theta(x)$ for each fixed $x\in\mathbb{R}^+$. Then when $\tau\in\Theta(x)$ by monotonicity of A in τ we obtain

$$A(x,\tau) \geq A(x,\Phi_{\eta_0}(x)) = \Phi_{\eta_0}(x),$$

$$A(x,\tau) \leq A(x,\eta) = \Phi_{\eta_0+\eta_1}(x)\frac{\eta}{\eta+\Phi_{\eta_1}(x)} \leq \frac{\eta(\eta_0+\eta_1)\int_x^\infty K(\tau)\,d\tau}{\eta_0+\eta_1+\Phi_{\eta_1}(x)}$$
$$\leq \eta\int_x^\infty K(\tau)\,d\tau = \Phi_\eta(x).$$

Since $A(x,\tau)$ is continuous by totality of arguments on set $\mathbb{R}^+\times[0,\eta]$, then it is obvious that $A(x,\tau)$ satisfies Caratheodory condition on $\mathbb{R}^+\times[0,\eta]$, in $\tau\in[0,\eta]$.

Analogously it is easy to check conditions i_1), i_2) for Examples 3.5–3.8.

3.3 Proof of theorems

First we prove Theorem 3.2. It follows from the definition of half infinite set Δ (see subsection 2.3) that

$$\gamma^* = \frac{\eta(1-\gamma_+)}{2} \in \Delta \tag{3.9}$$

Therefore results of subsection 2.3 imply the existence of positive and bounded solution $Q_{\gamma^*}(x)$ of equation (2.12), moreover

$$\lim_{x\to\infty} Q_{\gamma^*}(x) = 2\gamma^*(1-\gamma_+)^{-1} = \eta \tag{3.10}$$

and

$$S_{\gamma^*}(x) \le 2S_{\gamma^*}(x) - \psi_{\gamma^*}(x) \le Q_{\gamma^*}(x) \le 2S_{\gamma^*}(x), \quad x > 0 \tag{3.11}$$

Taking into account (2.11) and definition of function $S_{\gamma^*}(x)$ we have

$$S_{\gamma^*}(x) = \gamma^* S(x) \uparrow \gamma^*(1-\gamma_+)^{-1} = \frac{\eta}{2}, \quad x \to \infty. \tag{3.12}$$

Now we show that

$$\frac{\eta}{2} - S_{\gamma^*} \in L_1(0,+\infty). \tag{3.13}$$

If we denote by

$$\chi_{\gamma^*}(x) = \frac{\eta}{2} - S_{\gamma^*}(x). \tag{3.14}$$

then $\chi_{\gamma^*}(x) \ge 0$ satisfies the following equation

$$\chi(x) = \Phi_{\frac{\eta}{2}}(x) + \int_0^\infty K(x-t)\chi(t)\,dt, \quad x > 0. \tag{3.15}$$

Since kernel $K(x)$ possesses properties (1.2)–(1.3), then by Fubin's theorem (see [7]), it is easy to verity that

$$\Phi_{\frac{\eta}{2}} \in L_1(0,+\infty), \quad m_1(\Phi_{\frac{\eta}{2}}) \equiv \int_0^\infty x\Phi_{\frac{\eta}{2}}(x)\,dx < +\infty. \tag{3.16}$$

As $\Phi_{\frac{\eta}{2}}$ satisfies condition (3.16), and kernel possesses properties (1.2)-(1.4), then it follows from the result of [5] that equation (3.15) has a unique solution, which belongs to $L_1(0,+\infty) \cap L_\infty(0,+\infty)$.

Therefore $\chi_{\gamma^*} \in L_1(0,+\infty) \cap L_\infty(0,+\infty)$ is the unique solution of equation (3.15) in $L_1(0,+\infty) \cap L_\infty(0,+\infty)$. Since $\chi_{\gamma^*}, \psi_{\gamma^*} \in L_1(0,+\infty) \cap L_\infty(0,+\infty)$ from (3.11) we get

$$\eta - Q_{\gamma^*}(x) \le 2\chi_{\gamma^*}(x) + \psi_{\gamma^*}(x) \in L_1(0,+\infty) \cap L_\infty(0,+\infty)$$

$$\eta - Q_{\gamma^*}(x) \ge 2\chi_{\gamma^*}(x). \tag{3.17}$$

Therefore

$$\eta - Q_{\gamma^*}(x) \in L_1(0,+\infty) \cap L_\infty(0,+\infty). \tag{3.18}$$

$$\lim_{x\to\infty} Q_{\gamma^*}(x) = \eta. \tag{3.19}$$

Now we consider the following special iteration for basic equation (1.1).

$$\begin{gathered} \varphi_{n+1}(x) = A(x,\varphi_n(x)) + \int_0^\infty K(x-t)B(t,\varphi_n(t))\,dt \\ \varphi_0(x) = \Phi_{\eta_0}(x), \quad n = 0,1,2,\ldots, \quad x > 0 \end{gathered} \tag{3.20}$$

We are going to prove that

$$\begin{gathered} a)\quad \varphi_n(x)\!\uparrow \text{ by } n, \qquad b)\quad \Phi_{\eta_0}(x) \le \varphi_n(x) \le \eta - Q_{\gamma^*}(x) \\ n = 0, 1, 2, \ldots, \quad x > 0 \end{gathered} \tag{3.21}$$

We start with statement a). Indeed when $n = 0$ it is enough to prove that

$$\Phi_{\eta_0}(x) \le \eta - Q_{\gamma^*}(x).$$

We have

$$\eta - Q_{\gamma^*}(x) \ge 2\chi_{\gamma^*}(x) \ge \eta \int_x^\infty K(\tau)\, dt \ge \eta_0 \int_x^\infty K(\tau)\, d\tau = \Phi_{\eta_0}(x).$$

Assume that relation b) of (3.21) is true for any $n \in \mathbb{N}$. Then by monotonicity A and B we get

$$\varphi_{n+1}(x) \ge A(x, \Phi_{\eta_0}(x)) + \int_0^\infty K(x-t) B(t, \Phi_{\eta_0}(t))\, dt \ge \Phi_{\eta_0}(x).$$

Since

$$Q_{\gamma^*}(x) \le \eta, \quad Q_{\gamma^*}(x) \ge S_{\gamma^*}(x)$$

and

$$S_{\gamma^*}(x) \ge \gamma^* \ge \gamma_0 \ge A_0,$$

then using the properties of functions A, B and ω from (3.20) we obtain

$$\begin{aligned} \varphi_{n+1}(x) &\le A(x, \eta - Q_{\gamma^*}(x)) + \int_0^\infty K(x-t) B(t, \eta - Q_{\gamma^*}(t))\, dt \\ &\le A(x, \eta) + \int_0^\infty K(x-t)(\eta - Q_{\gamma^*}(t) + \omega(t, \eta))\, dt \\ &\le \Phi_\eta(x) + \eta \int_{-\infty}^x K(\tau) d\tau - \int_0^\infty K(x-t)(Q_{\gamma^*}(t) - \omega(t, \eta))\, dt \\ &\le \eta - \int_0^\infty K(x-t)(Q_{\gamma^*}(t) - \omega(t, Q_{\gamma^*}(t)))\, dt = \eta - Q_{\gamma^*}(x). \end{aligned}$$

Proof of statement a) (see (3.21)) is more simple. Therefore the sequence of functions $\{\varphi_n(x)\}_{n=0}^{+\infty}$ almost everywhere has the pointwise limit in $(0, +\infty)$

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \Phi_{\eta_0}(x) \le \varphi(x) \le \eta - Q_{\gamma^*}(x). \tag{3.22}$$

From B. Levi's theorem (see [7]) we conclude that the limit function satisfies equation (1.1). Thus the Theorem 3.2 is proved.

If we consider equation (2.9) and instead of (3.22) use the following double inequality

$$\Phi_{\eta_0}(x) \le \varphi(x) \le \eta - \eta(1 - \gamma_+) S(x) \in L_1(0, +\infty) \cap L_\infty(0, +\infty) \tag{3.23}$$

then the proof of the Theorem 3.1 can be realized analogously.

Remark 3.2 As known (see [8]) in case where $A(x,\tau)\equiv 0$, $B(t,z)=z$ under the conditions (1.2)–(1.4) equation (1.1) in space of slow growth functions has only monotonic increasing and bounded solution. Note that new conditions on nonlinearity describing functions $A(x,\tau)$ and $B(x,\tau)$ reduce to qualitative changes of behavior of solutions (see Theorems 3.1 and 3.2).

4 Continuity and uniqueness of solution in one particular case

4.1 Uniqueness of solution

Let functions $A(x,r)$ and $B_0(z)$ be given by (3.8) and (3.6). Then equation (1.1) has the from

$$\varphi(x)=\Phi_{\eta_0+\eta_1}(x)\frac{\varphi(x)}{\varphi(x)+\Phi_{\eta_1}(x)}+\frac{1}{\eta^{p-1}}\int_0^\infty K(x-t)\varphi^p(t)\,dt \tag{4.1}$$

$$p\geq 2,\quad \eta_0,\eta_1>0,\quad \eta\geq\eta_0+\eta_1,\quad x>0. \tag{4.2}$$

The following theorem holds

Theorem 4.1 *Let $\eta\geq\eta_0+\eta_1$, $\eta_0,\eta_1>0$, $p\geq 2$, $\gamma_+<\frac{1}{2}$. In addition, we assume that kernel $K(z)$ possesses properties* (1.2)–(1.4)*. If $\frac{\eta_0}{\eta_0+\eta_1}>p\gamma_+^{p-1}$ then equation* (4.1) *has a unique solution in class*

$$\mathfrak{M}\equiv\{f(x)\colon f(x)\ \text{measurable and}\ \Phi_{\eta_0}(x)\leq f(x)\leq\eta\gamma_+\}.$$

Proof Let $\varphi_1,\varphi_2\in\mathfrak{M}$ be two different solutions of equation (4.1) in the class $\mathfrak{M}$. We have

$$\varphi_1(x)-\varphi_2(x)=\Phi_{\eta_0+\eta_1}(x)\left(\frac{\varphi_1(x)}{\varphi_1(x)+\Phi_{\eta_1}(x)}-\frac{\varphi_2(x)}{\varphi_2(x)+\Phi_{\eta_1}(x)}\right)$$

$$+\frac{1}{\eta^{p-1}}\int_0^\infty K(x-t)(\varphi_1^p(t)-\varphi_2^p(t))\,dt. \tag{4.3}$$

As $\varphi_1,\varphi_2\in\mathfrak{M}$ then

$$|\varphi_1^p-\varphi_2^p|\leq p\eta^{p-1}\gamma_+^{p-1}|\varphi_1-\varphi_2|. \tag{4.4}$$

Taking into account (4.4) and the fact $\varphi_1,\varphi_2\in\mathfrak{M}$, from (4.3) we get

$$|\varphi_1(x)-\varphi_2(x)|\leq\Phi_{\eta_0+\eta_1}(x)\Phi_{\eta_1}(x)\frac{|\varphi_1(x)-\varphi_2(x)|}{(\varphi_1(x)+\Phi_{\eta_1}(x))(\varphi_2(x)+\Phi_{\eta_1}(x))}$$

$$+p\gamma_+^{p-1}\int_0^\infty K(x-t)|\varphi_1(t)-\varphi_2(t)|dt\leq\frac{\eta_1}{\eta_0+\eta_1}|\varphi_1(x)-\varphi_2(x)|$$

$$+p\gamma_+^{p-1}\int_0^\infty K(x-t)|\varphi_1(t)-\varphi_2(t)|dt$$

$$\leq\left(\tfrac{\eta_1}{\eta_0+\eta_1}+p\gamma_+^{p-1}\right)\operatorname*{esssup}_{x\in\mathbb{R}^+}|\varphi_1(x)-\varphi_2(x)|. \tag{4.5}$$

It follows from results of [4] that if $p \geq 2$, $0 < \gamma_+ < \frac{1}{2}$, then

$$p\gamma_+^{p-1} < 1. \tag{4.6}$$

From (4.5) we get

$$\left(\frac{\eta_0}{\eta_0 + \eta_1} - p\gamma_+^{p-1}\right) \operatorname*{esssup}_{x \in \mathbb{R}^+} |\varphi_1(x) - \varphi_2(x)| \leq 0. \tag{4.7}$$

Since $\frac{\eta_0}{\eta_0+\eta_1} \in (p\gamma_+^{p-1}, 1)$, then from (4.7) we obtain $\varphi_1(x) = \varphi_2(x)$ almost everywhere in $(0, +\infty)$. Thus the proof is complete. □

4.2 Continuity of solution

In this subsection we prove that the obtained solution of (1.1) is also continuous.

Theorem 4.2 *Assume all conditions of theorem* (4.1) *are fulfilled. Then solution of equation* (4.1) *belongs to space* $L_1(0, +\infty) \cap C_M(0, +\infty)$.

Proof As $S(x) \geq S(0) = 1$ then from (3.23) we obtain

$$\Phi_{\eta_0}(x) \leq \eta - \eta(1 - \gamma_+)S(x) \leq \eta\gamma_+. \tag{4.8}$$

Now we consider iteration (3.20) for equation (4.1). By induction we show that the following inequality holds

$$|\varphi_{n+1}(x) - \varphi_n(x)| \leq \eta\gamma_+^p\rho^n, \quad n = 0, 1, 2, \ldots, \tag{4.9}$$

where

$$\rho \equiv \frac{\eta_1}{\eta_0 + \eta_1} + p\gamma_+^{p-1}. \tag{4.10}$$

If $n = 0$ from (3.20), (4.1) and (4.8) we obtain

$$|\varphi_1(x) - \varphi_0(x)| = \frac{1}{\eta^{p-1}} \int_0^\infty K(x - t)\varphi_0^p(t)\, dt \leq \frac{\eta^p\gamma_+^p}{\eta^{p-1}} = \eta\gamma_+^p.$$

Assume that (4.9) is true for any $n \in \mathbb{N}$. Then using (3.20), (4.4) and $\varphi_n(x) \geq \Phi_{\eta_0}(x)$, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} |\varphi_{n+2}(x) - \varphi_{n+1}(x)| &\leq \frac{\eta_1}{\eta_0 + \eta_1}|\varphi_{n+1}(x) - \varphi_n(x)| \\ &+ \frac{1}{\eta^{p-1}} \int_0^\infty K(x - t)|\varphi_{n+1}^p(t) - \varphi_n^p(t)|\, dt \\ &\leq \left(\tfrac{\eta_1}{\eta_0+\eta_1} + p\gamma_+^{p-1}\right) \eta\gamma_+^p, \quad \rho^n = \eta\gamma_+^p\rho^{n+1}. \end{aligned} \tag{4.11}$$

Thus in accordance with Weierstrass theorem it follows from (4.11) that the convergence of sequence of functions $\{\varphi_n(x)\}_{n=0}^\infty$ is uniform.

By induction, it is easy to see that $\varphi_n \in C[0, +\infty)$. Therefore, the well known theorem from calculus yields continuity of limit function $\varphi(x)$ on $[0, +\infty)$. Taking into account Theorem 3.1, we get complete the proof. □

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