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Two-sided Tolerance Intervals in a Simple Linear Regression*

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Abstract

Numerical results for a simple linear regression indicate that the non-simultaneous two-sided tolerance intervals nearly satisfy the condition of multiple-use confidence intervals, see Lee and Mathew (2002), but the numerical computation of the limits of the multiple-use confidence intervals is needed. We modified the Lieberman–Miller method (1963) for computing the simultaneous two-sided tolerance intervals in a simple linear regression with independent normally distributed errors. The suggested tolerance intervals are the narrowest of all the known simultaneous two-sided tolerance intervals. The computation of the multiple-use confidence intervals based on the new simultaneous two-sided tolerance intervals is simple and fast.

Key words: multiple-use confidence interval, simultaneous two-sided tolerance interval

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1 Introduction

In this article we assume the two-sided tolerance intervals in a simple linear regression model with independent normally distributed errors. The progress of the methods for determining the simultaneous two-sided tolerance intervals in a

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linear regression is motivated by the task of a univariate multiple-use calibration, it has been exploited by several authors to solve the problem of constructing the multiple-use confidence intervals by inverting the simultaneous two-sided tolerance intervals, see e.g. [7], [10]. A tolerance interval is specified by its content denoted γ and a confidence level denoted $1 - \alpha$. In practical applications the values γ, $1 - \alpha$ are close to one. It is desired to obtain the narrowest simultaneous two-sided tolerance intervals subject to the confidence and the content requirements, resulting in the narrowest multiple-use confidence intervals. For the case of a simple linear regression a numerical results provided in [5] and in [9] indicate that the multiple-use confidence intervals can be obtained by inverting the non-simultaneous two-sided tolerance intervals. In the case of a fixed value of explanatory variable, the results for determining a tolerance interval for a normal distribution can be applied for computing the non-simultaneous two-sided tolerance interval for a linear regression, see e.g. [4].

The various ways to construct the simultaneous two-sided tolerance intervals in a general form have been described in literature. One way to determine the width of the simultaneous two-sided tolerance interval is based on a specified general confidence set for unknown parameters of the model. The coverage of the simultaneous two-sided tolerance intervals constructed by the methods derived for different confidence sets for parameters of the model was investigated for a simple linear regression model in [1]. The confidence level exceeds the nominal value in all the methods, the empirical probability of coverage of the simultaneous two-sided tolerance intervals constructed by the second Chvosteková method [1] is the least conservative. In the Lieberman–Miller method [6] and in the Mee–Eberhardt–Reeve method [10] the width of the simultaneous two-sided tolerance intervals is expressed in a certain simple functional form. The confidence level of the simultaneous two-sided tolerance intervals constructed by these two methods is approximately equal to the nominal level. Only the coverage of the simultaneous two-sided tolerance intervals constructed by the Witkovský method [13] is equal to the nominal level exactly. The exact simultaneous two-sided tolerance intervals for a linear regression are computed by a generalization of the method for computing the simultaneous two-sided tolerance intervals for several independent normal populations. While the non-simultaneous two-sided tolerance intervals are narrower than all the known simultaneous tolerance intervals, the computation of the multiple-use confidence intervals is easier with the simultaneous two-sided tolerance intervals constructed by the Mee–Eberhardt–Reeve method and by the Lieberman–Miller method.

Note that Scheffé [11] assumed the simultaneous equail-tailed tolerance intervals (the content requirement is related to the center of the response variable distribution), but the requirement of central tolerance intervals is not necessary to derive the confidence intervals for calibration. We do not deal with these tolerance intervals here. The exact simultaneous equal-tailed tolerance intervals for the considered model can be found in [3].

In the case of a simple linear regression with independent, normally distributed errors we investigated the coverage of the simultaneous two-sided tolerance intervals constructed by the Mee–Eberhardt–Reeve method, by the Lieberman–Miller method, and based on the non-simultaneous tolerance intervals. The simultaneous two-sided tolerance intervals determined by the suggested modified Lieberman–Miller method are the narrowest. Although the empirical probability of coverage of the simultaneous two-sided tolerance intervals computed by the modified Lieberman–Miller method is below the nominal level, the simultaneous two-sided tolerance intervals are suitable for application in multiple-use calibration problem.

2 Two-sided tolerance intervals in the underlying model

We consider a normal simple linear regression model

$$Y_i = \beta_0 + \beta x_i + \epsilon_i, \epsilon_i \sim N(0, \sigma^2), i = 1, \ldots, n, \tag{2.1}$$

where Y_i are independent, normally distributed observations, x_i are the known values of an explanatory variable, $i = 1, \ldots, n$, $\beta^T = (\beta_0, \beta_1)$ and σ^2 are the unknown parameters of the model. Let $\hat{\beta}$ denote the least squares estimator of β, and let S^2 denote the residual mean square based on $n - 2$ degrees of freedom. Without loss of generality we assume $\bar{x} = \sum_{i=1}^n x_i/n = 0$ and under the assumption for the model (2.1) it holds

$$\begin{aligned} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} &= \begin{pmatrix} \bar{Y} \\ \frac{S_{xY}}{S_{xx}} \end{pmatrix} \sim N_2\left(\beta, \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{S_{xx}} \end{pmatrix}\right), \\ \frac{(n-2)S^2}{\sigma^2} &= \frac{\sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{\sigma^2} \sim \chi^2_{n-2}, \end{aligned} \tag{2.2}$$

where

$$\bar{Y} = \sum_{i=1}^n Y_i/n, \quad S_{xY} = \sum_{i=1}^n x_i(Y_i - \bar{Y}), \quad S_{xx} = \sum_{i=1}^n x_i^2,$$

χ^2_{n-2} denotes a central chi-square random variable with $n-2$ degrees of freedom and random variables $\hat{\beta}$ and S^2 are independent.

Let $Y(x)$ denote a future observation of a response at a value x, then $Y(x) = \beta_0 + \beta_1 x + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$ and $Y(x)$ is assumed to be independent of a vector of observations $Y = (Y_1, \ldots, Y_n)$. The (general) form of a two-sided $(\gamma, 1-\alpha)$-tolerance interval for distribution of $Y(x)$ corresponding to x is $[\hat{\beta}_0 + \hat{\beta}_1 x - \lambda(x)S, \hat{\beta}_0 + \hat{\beta}_1 x + \lambda(x)S]$, where $\lambda(x)$ is a *tolerance factor* to be determined subject to the content and the confidence level requirements.

The non-simultaneous two-sided tolerance interval is constructed based on the vector of observations such that it contains at least γ proportion of the $Y(x)$-distribution with confidence $1 - \alpha$. In the case of a fixed value of explanatory variable the *non-simultaneous* tolerance factor is determined by using the results for computation of a tolerance factor for a normal distribution, see e.g. [4].

The simultaneous two-sided tolerance interval for the distribution of $Y(x)$ corresponding to possible different values of x is constructed based on the same estimated regression line such that with confidence $1-\alpha$ the minimum content (with respect to x) is γ. Hence, a function $\lambda(.)$, the *simultaneous* tolerance factor, satisfies the condition

$$P_{\hat{\beta},S}\left(\min_{x\in R} P_{Y(x)}(\hat{\beta}_0+\hat{\beta}_1 x-\lambda(x)S\leq Y(x)\leq \hat{\beta}_0+\hat{\beta}_1 x+\lambda(x)S \mid \hat{\beta},S)\geq\gamma\right) = 1-\alpha. \tag{2.3}$$

Let $C(x;\hat{\beta},S)$ denote the content of the simultaneous two-sided tolerance interval at an x conditionally given $\hat{\beta},S$, i.e. $C(x;\hat{\beta},S)=P_{Y(x)}(\hat{\beta}_0+\hat{\beta}_1 x-\lambda(x)S\leq Y(x)\leq \hat{\beta}_0+\hat{\beta}_1 x+\lambda(x)S \mid \hat{\beta},S)$. Then, by straightforward calculations, we get

$$C(x;\hat{\beta},S)=\Phi\left(\frac{B_0}{\sqrt{n}}+\frac{x}{\sqrt{S_{xx}}}B_1+\lambda(x)U\right)-\Phi\left(\frac{B_0}{\sqrt{n}}+\frac{x}{\sqrt{S_{xx}}}B_1-\lambda(x)U\right) = C(x;B,U), \tag{2.4}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution, $B_0=\sqrt{n}(\bar{Y}-\beta_0)/\sigma\sim N(0,1)$ and $B_1=\sqrt{S_{xx}}(\hat{\beta}_1-\beta_1)/\sigma\sim N(0,1)$ are independently distributed, $U=S/\sigma$, $(n-2)U^2\sim\chi^2_{n-2}$.

Let $c=x/\sqrt{S_{xx}}$, we shall use the notation $\lambda(c)$ instead of $\lambda(x)$ and $C(c,B,U)$ instead of $C(x,B,U)$. Thus

$$C(c,B,U)=\Phi\left(\frac{B_0}{\sqrt{n}}+cB_1+\lambda(c)U\right)-\Phi\left(\frac{B_0}{\sqrt{n}}+cB_1-\lambda(c)U\right). \tag{2.5}$$

Hence, the condition (2.3) can be written as

$$P_{B,U}\left(\min_{c\in R} C(c;B,U)\geq\gamma\right)=1-\alpha. \tag{2.6}$$

The notation for the condition (2.6) is taken from [5].

The non-simultaneous tolerance factor and the simultaneous tolerance factors in all the known methods depend on c through the value $d(c)$. For a simple linear regression $d(c)=\sqrt{1/n+c^2}$, thus the tolerance factors are symmetric in c around zero, $d(c)=d(-c)$. Note that the simultaneous tolerance factors in the Mee–Eberhardt–Reeve method (MER) and in the Witkovský method (eW) are determined subject to the confidence and the content requirement with respect to $c\in I=[c_{min},c_{max}]$. The assumption $|c|\leq 1$ (i.e. $|x|\leq\sqrt{S_{xx}}$) is reasonable for the calibration problem, see [5]. It is also possible to determine the simultaneous tolerance factor subject to $c\in I$ in the Lieberman–Miller method (LM). A simultaneous tolerance factor in the methods based on the 'general confidence set approach' (GCSA methods, see [1]) is derived under the assumption $c\in R$.

The coverage of the simultaneous two-sided tolerance intervals constructed by the GCSA methods was investigated for a simple linear regression model in [1]. The confidence level exceeds the nominal value in all the GCSA methods, the empirical probability of coverage of the simultaneous two-sided tolerance intervals constructed by the second Chvosteková method [1] is the least conservative. Fig. 1. shows the simultaneous tolerance factors computed by the GCSA methods, by the Wilson method (W) [12], by the modified Wilson method (MW) [8], by the Limam–Thomas method (LT) [8], by the Chvosteková method (C) [2] and by the second Chvosteková method (C-II) [1] for $\alpha = 0.05$, $\gamma = 0.95$, $n = \{10, 20\}$.

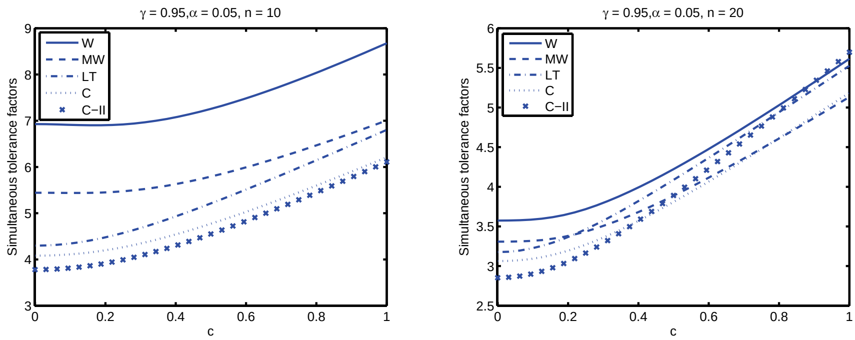

Figure 1: The simultaneous tolerance factors evaluated by the GCSA methods for $\alpha = 0.05$, $\gamma = 0.95$ and $n = \{10, 20\}$ in the simple linear regression.

The simultaneous tolerance factors computed by the second Chvosteková method are the lowest over a range of possible values of c for $n = 10$, while in the case $n = 20$ they are the largest for the c's close to 1.

The simultaneous two-sided tolerance intervals constructed by the GCSA methods are all derived for unlimited range of values c and in addition their confidence level exceeds the nominal level, so they are possibly wider than it is needed for use in statistical calibration. For the case of a simple linear regression a numerical results provided in [5] and in [9] indicate that the multiple-use confidence intervals can be obtained by inverting the non-simultaneous two-sided tolerance intervals. The repeated extensive calculations are demanded to determine the non-simultaneous tolerance factors and also to determine the simultaneous tolerance factors computed by the eW for each possible value of c. The advantage of the LM and the MER is expression of the simultaneous tolerance factor in a simple functional form, $\lambda(c) = \lambda l(d(c))$, where $l(d(c))$ is a certain function of c and the different procedures to compute a scalar λ to satisfy the condition (2.6) for given $\gamma, 1-\alpha, n$ are proposed in the MER and in the LM. In the MER, $l_{\mathrm{MER}}(c) = u((1+\gamma)/2)+2d(c)$, where $u((1+\gamma)/2)$ denotes a $(1+\gamma)/2$-quantile of standard normal distribution, while in the LM a simpler form is proposed, $l_{LM}(c) = d(c)$. For the example with $\alpha = 0.05$, $\gamma = 0.95$, $|c| \leq 1$, the values of λ computed by the LM, denoted λ_{LM}, are 11.161 and 12.447 for $n = 10$ and $n = 20$, respectively. For the same inputs, the values of λ computed by the MER, denoted λ_{MER}, are 1.469 and 1.239 for $n = 10$ and $n = 20$, respectively. Fig. 2 shows the non-simultaneous tolerance factors (NSTF), the simultaneous tolerance factors computed by the C-II, and by the LM, by the MER, by the eW under assumption $|c| \leq 1$ for $\alpha = 0.05$, $\gamma = 0.95$, $n = \{10, 20\}$.

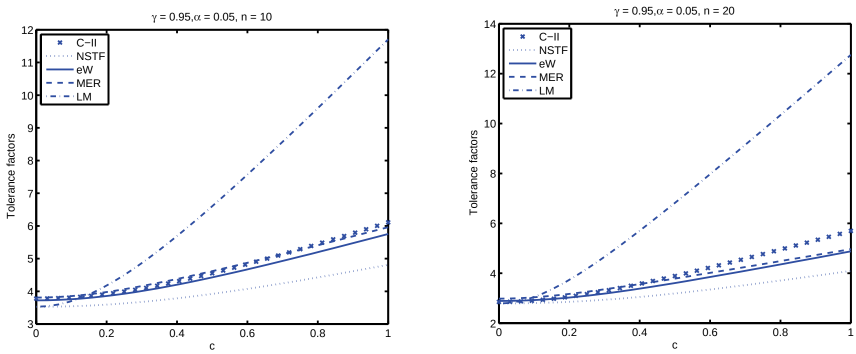

Figure 2: The non-simultaneous tolerance factors (NSTF) and the simultaneous tolerance factors evaluated by the C-II, by the LM, by the MER, by the eW for $\alpha = 0.05$, $\gamma = 0.95$ and $n = \{10, 20\}$ in the simple linear regression.

The values of the simultaneousity parameter $\widetilde{m}$ for eW (see [13]) are 3.6 and 4.2 for $n = 10$ and $n = 20$, respectively. The non-simultaneous tolerance factors are the lowest over a range of possible values of c, at c's close to the 0 the simultaneous tolerance factors computed by the LM are close to the values of the non-simultaneous tolerance factors. The confidence level of the simultaneous two-sided tolerance intervals constructed by the eW is equal to the nominal level. Because the simultaneous tolerance factors determined by the MER are higher than the simultaneous tolerance factors determined by the eW over the range of values c in both cases $n = 10$, $n = 20$, it can be expected that the confidence level of the simultaneous tolerance intervals constructed by the MER is over the nominal level. The simultaneous two-sided tolerance intervals constructed by the C-II are conservative. For the case $n = 10$ the simultaneous tolerance factors computed by the C-II are higher than the simultaneous tolerance factors determined by the eW over the range $|c| < 1$ and the difference between the simultaneous tolerance factors increases with increasing values of $|c|$. For the case $n = 20$ the simultaneous tolerance factors determined by the C-II are lower than the simultaneous tolerance factors determined by the eW for $c < 0.1$, for c's close to 1 the difference between the simultaneous tolerance factors is larger than in the case $n = 10$.

Since the simultaneous two-sided tolerance factors computed by the LM method are narrower than the simultaneous two-sided tolerance factors computed by the MER method for c's close to 0 but they are markedly wider with increasing values of $|c|$, we suggested to determine the scalar value λ by the LM with $l(d(c)) = l_{\mathrm{MER}}(d(c))$. Hence, λ computed by the modified LM method (mLM) is given as

$$\lambda_{\mathrm{mLM}} = h\sqrt{(n-2)/\chi^2_{n-2}(\alpha)}, \tag{2.7}$$

where $\chi^2_{n-2}(\alpha)$ denotes an α-quantile of the chi-square distribution with $n-2$ degrees of freedom. Readers are referred to the Lieberman–Miller paper [6] for details. In the suggested method, a value h is found by a numerical method to satisfy the equation

$$\min_{c \in I} \left[\Phi\left(\frac{1}{\sqrt{n}} + c + hl_{\mathrm{MER}}(d(c)) \right) - \Phi\left(\frac{1}{\sqrt{n}} + c - hl_{\mathrm{MER}}(d(c)) \right) \right] = \gamma. \quad (2.8)$$

For $\alpha = 0.05$, $\gamma = 0.95$, $|c| \leq 1$, the values of λ_{mLM} are 1.386 and 1.164 for $n = 10$ and $n = 20$, respectively. The simultaneous tolerance factor in the mLM and in the MER is expressed in the same form

$$\lambda(c) = \lambda[u((1+\gamma)/2) + 2\sqrt{1/n + c^2}].$$

Since it holds $\lambda_{\mathrm{mLM}} < \lambda_{\mathrm{MER}}$ for these settings, it is clear that the simultaneous tolerance factors computed by the mLM are narrower than the simultaneous tolerance factors computed by the MER. Fig. 3 shows the simultaneous tolerance factors computed by the MER, by the eW, by the mLM and the non-simultaneous tolerance factors.

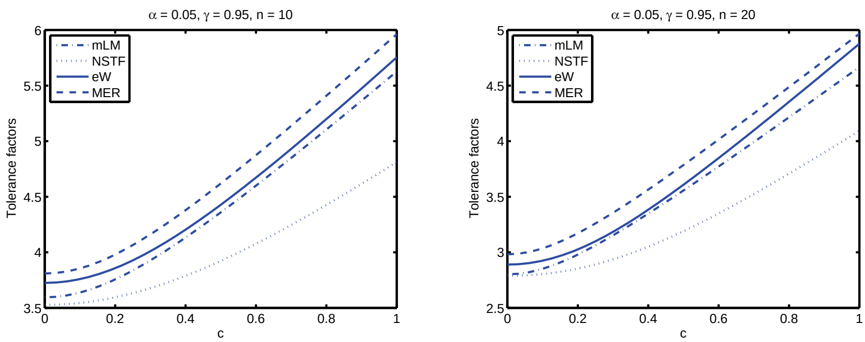

Figure 3: The simultaneous tolerance factors evaluated by the MER, by the eW, by the mLM and the non-simultaneous tolerance factors (NSTF) for $\alpha = 0.05$, $\gamma = 0.95$ and $n = \{10, 20\}$ in the simple linear regression.

The difference of the simultaneous tolerance factors computed by the mLM and by the MER is equal to $(\lambda_{\mathrm{MER}} - \lambda_{\mathrm{mLM}})[u((1+\gamma)/2) + 2\sqrt{1/n + c^2}]$ and the function $2(\lambda_{\mathrm{MER}} - \lambda_{\mathrm{mLM}})\sqrt{1/n + c^2}$ of c increases slowly with increasing values of $|c|$, the simultaneous tolerance factors computed by the mLM seem to be parallel with the simultaneous tolerance factors computed by the MER. The simultaneous tolerance factors determined by the eW are higher than the simultaneous tolerance factors determined by the mLM over the range of values c in both case $n = 10$, $n = 20$, the largest difference between the values of the simultaneous tolerance factors is at the points 0 and 1. The simultaneous two-sided tolerance intervals constructed by the mLM are the narrowest, but their confidence level below the nominal level can be expected.

In the next section we numerically investigate the coverage of the simultaneous two-sided tolerance intervals constructed by the LM, by the MER, by the mLM and based on the non-simultaneous tolerance factor for $|c| \leq 1$ in the case of a simple linear regression.

3 A simulation study

We provide a simulation study of the simultaneous two-sided tolerance intervals in the form (2.6). For the case of a simple linear regression model it holds $(n-2)U^2 \sim \chi^2_{n-2}$, $B_0 \sim N(0,1)$ and $B_1 \sim N(0,1)$. We generated $N = 10000$ triples (b_0, b_1, u) of the random variables B_0, B_1, U and for each sample we computed the $\min_{|c|\le 1} C(c; b_0, b_1, u)$, where the simultaneous tolerance factor was computed by the LM, by the MER, by the mLM and as the non-simultaneous tolerance factor (NSTF). Tab. 1 shows the empirical probabilities of coverage of the simultaneous two-sided tolerance intervals constructed by the mentioned methods for combinations of the content $\gamma = \{0.90, 0.95, 0.99\}$, the confidence level $1-\alpha = \{0.90, 0.95, 0.9\}$ and $n = \{10, 20\}$.

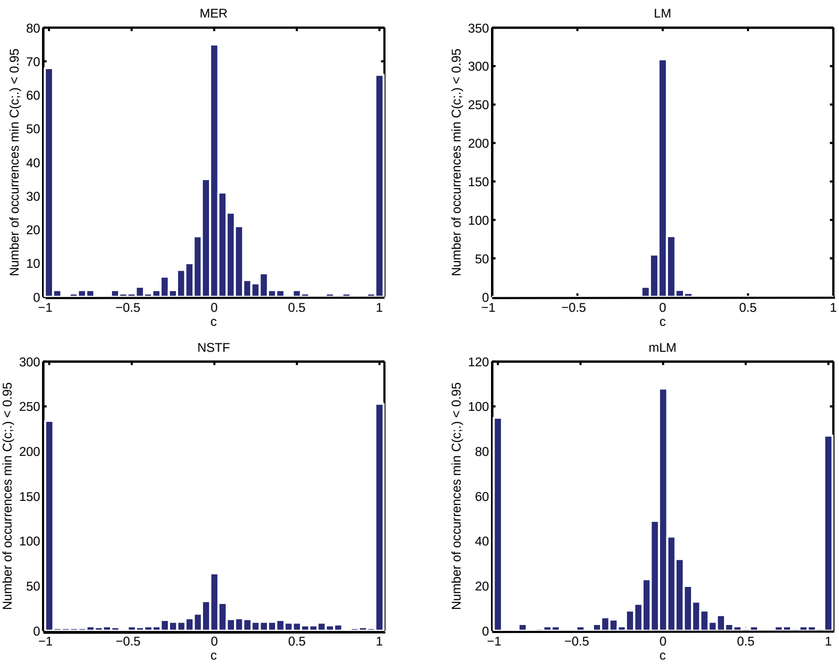

Figure 4: The histograms of the samples for which the minimal content was under the γ for the $(0.95, 0.95)$ simultaneous two-sided tolerance intervals computed by the MER, by the LM, by the mLM and based on the NSTF.

The empirical probabilities of coverage of the simultaneous two-sided tolerance intervals constructed by the MER are larger than the prescribed confidence level as we expected. The simultaneous two-sided tolerance intervals constructed by the LM satisfy the condition (2.6) quite well. The empirical probabilities of coverage of the simultaneous two-sided tolerance intervals determined by the suggested method is closer to the prescribed nominal level for $n = 10$ than for $n = 20$, but it is always below the stated value. In [9] it was indicated that for $\alpha = 0.01$ and $\gamma = 0.95$, the non-simultaneous tolerance factor instead of a simultaneous tolerance factor can be used to construct the simultaneous two-sided tolerance intervals. Based on the simulation results, the empirical coverage probability of the simultaneous two-sided tolerance intervals constructed based on the non-simultaneous tolerance factor is below the nominal level, in general.

Method	n	$\gamma = 0.90$		
		$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
LM	10	0.9034	0.9521	0.9916
MER		0.9202	0.9607	0.9927
NSTF		0.8305	0.9079	0.9805
mLM		0.8864	0.9415	0.9873
LM	20	0.8948	0.9496	0.9893
MER		0.9222	0.9624	0.9933
NSTF		0.8013	0.8940	0.9760
mLM		0.8714	0.9291	0.9821
Method	**n**	$\gamma = 0.95$		
		$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
LM	10	0.9048	0.9525	0.9917
MER		0.9172	0.9592	0.9928
NSTF		0.8368	0.9123	0.9822
mLM		0.8921	0.9446	0.9887
LM	20	0.8951	0.9457	0.9891
MER		0.9202	0.9602	0.9931
NSTF		0.8061	0.8933	0.9744
mLM		0.8749	0.9324	0.9827
Method	**n**	$\gamma = 0.99$		
		$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
LM	10	0.9062	0.9536	0.9923
MER		0.9150	0.9573	0.9927
NSTF		0.8470	0.9200	0.9831
mLM		0.8984	0.9487	0.9906
LM	20	0.8961	0.9490	0.9899
MER		0.9139	0.9608	0.9931
NSTF		0.8147	0.9009	0.9777
mLM		0.8814	0.9367	0.9859

Table 6: The empirical probabilities of coverage of the simultaneous two-sided tolerance intervals constructed by the LM, by the MER, by the mLM and by the NSTF under the assumption $|c| < 1$ for combinations of the content $\gamma = \{0.90, 0.95, 0.99\}$, the confidence level $\alpha = \{0.10, 0.05, 0.01\}$ and $n = \{10, 20\}$.

The histograms in Fig. 4 display the number of the samples from the simulation study, for which the minimum of the content (2.5) was under γ and the corresponding argument of the minimum. The results are presented for the case $\alpha = 0.05$, $\gamma = 0.95$ and $n = 10$.

The simultaneous two-sided tolerance intervals constructed by the LM are close to non-simultaneous two-sided tolerance intervals for c's around zero and they markedly increase with increasing values of $|c|$. So, it is expected that the minimal content is solely achieved for c's close to zero.

Let $c^a = \arg\min_{|c|<1} C(c;.)$ and $C(c^a;.) < \gamma$ where the simultaneous two-sided tolerance interval is computed by the LM, then it holds $C(c^a;.) < \gamma$ also for the non-simultaneous two-sided tolerance interval. However, we can see in Fig. 4 that the minimal content of the non-simultaneous two-sided tolerance intervals is more often under the required level at c's close to -1 and 1. The simultaneous tolerance factors determined by the mLM are larger than the non-simultaneous tolerance factors for all possible values of c, the difference between the tolerance factors is larger with increasing values of $|c|$, see Fig. 3. The $\min_{|c|<1} C(c;.) < 0.95$ for the simultaneous tolerance interval constructed by the mLM occurred no so often at c's close to -1 and 1 as for the non-simultaneous tolerance interval, but the number of occurrence of $\min_{|c|<1} C(c;.) < 0.95$ at c's close to 0 is higher for the simultaneous tolerance interval constructed by the mLM. The difference between the simultaneous tolerance factors determined by the eW and by the mLM is the highest at points -1, 0, 1. The $\min_{|c|<1} C(c;.) < 0.95$ for the simultaneous tolerance interval constructed by the mLM occurred most frequently at points -1, 0, 1, the number of the occurrences at $-1, 0, 1$ is almost the same. The shape of the histograms for the simultaneous tolerance intervals constructed by the mLM and by the MER is similar. The empirical probability of the coverage of the non-simultaneous tolerance intervals and the simultaneous tolerance intervals constructed by the mLM for $n = 20$ is smaller than it is for $n = 10$, the number of occurrences of the minimal content under the prescribed γ at 0 is larger.

The empirical probability of the coverage of the simultaneous two-sided tolerance intervals constructed by the mLM is below the nominal level, the different functions $l(.)$ can be considered to achieve the better efficiency. Although the empirical coverage of the simultaneous two-sided tolerance intervals based on the non-simultaneous tolerance factor is below the prescribed level, the numerical results provided for the simple linear regression indicated that the non-simultaneous two-sided tolerance intervals nearly satisfy the property that arises in multiple-use calibration, see [5]. Note that the condition of the simultaneous two-sided tolerance intervals is sufficient for the condition of the multiple-use confidence intervals. The simultaneous two-sided tolerance intervals constructed by the mLM are wider than the non-simultaneous two-sided tolerance intervals, but the determining of the limits of the multiple-use confidence intervals is simpler and faster, for the computation formula see [9]. We assume that the mLM are suitable for application in the multiple-use calibration problems, however, further research about the multiple-use confidence intervals determined based on mLM is needed.

4 Conclusions

We suggested a procedure for computing the simultaneous two-sided tolerance intervals. The derived simultaneous two-sided tolerance intervals are the narrowest from all known simultaneous tolerance intervals. The suggested method was presented for the case of a simple linear regression, but the technique can be extended to a multiple regression model. The computation of the multiple-use confidence intervals by inverting the newly constructed simultaneous two-sided tolerance intervals is faster and simpler than by inverting the non-simultaneous two-sided tolerance intervals.

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