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Vector Optimization Results for ℓ-Stable Data*

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Abstract

The aim of this paper is to summarize basic facts about ℓ-stable at a point vector functions and existing results for certain vector constrained programming problem with ℓ-stable data.

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1 Introduction

In 2008 the concept of ℓ-stable at a point scalar functions was introduced in [1] as a generalization of $C^{1,1}$ functions—functions with locally Lipschitz derivative. The main aim was to receive more general optimality conditions than for $C^{1,1}$ functions which were extensively studied previously (see e.g. [8]). In subsequent years the attention was devoted to deriving other properties of ℓ-stable at a point functions and to extending ℓ-stability to finite–dimensional spaces in connection with vector optimization ([2, 3, 4, 5, 6, 7]).

In this paper, I try to summarize the most important of this existing results. The basic facts concerning vector ℓ-stability are recalled in the following section. Section 3 informs about second-order necessary and sufficient conditions for the following programming problem:

$$\text{minimize } f(x) \quad \begin{array}{l}\text{subject to } C \\ \text{such that } g(x) \in -K,\end{array} \tag{P}$$

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where $f\colon \mathbb{R}^N \to \mathbb{R}^M$ and $g\colon \mathbb{R}^N \to \mathbb{R}^P$, $M \in \mathbb{N}$, $N \in \mathbb{N}$, $P \in \mathbb{N}$, are given functions and $C \subset \mathbb{R}^M$, $K \subset \mathbb{R}^P$, are closed, convex, and pointed cones with non-empty interior (for definitions see e.g. [9]).

2 ℓ-stability

First of all I recall several fundamental notations which are used in this paper. The Euclidean norm and the scalar product in $\mathbb{R}^N$ are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, the zero element and the unit sphere of $\mathbb{R}^N$, i.e. the set $\{x \in \mathbb{R}^N; \|x\| = 1\}$, by $0_{\mathbb{R}^N}$ and $S_{\mathbb{R}^N}$, respectively. For a function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ and a point $x \in \mathbb{R}^N$, the symbol $f'(x)$ means the Fréchet derivative of f at x.

The scalar ℓ-stable at a point function was introduced in [1] using a lower directional derivative:

$$f^{\ell}(x;h) = \liminf_{t\downarrow 0} \frac{f(x+th)-f(x)}{t}$$

for $f\colon \mathbb{R}^N \to \mathbb{R}$, $x \in \mathbb{R}^N$, $h \in \mathbb{R}^N$.

Definition 2.1 We say that a function $f\colon \mathbb{R}^N \to \mathbb{R}$ is *ℓ-stable at* $x_0 \in \mathbb{R}^N$ if there are a neighbourhood U of x_0 and a constant $K > 0$ such that

$$|f^{\ell}(x;h) - f^{\ell}(x_0;h)| \le K\|x - x_0\|, \quad \forall x \in U,\ \forall h \in S_{\mathbb{R}^N}.$$

Following example presents a scalar function which is ℓ-stable at a point, but not differentiable on any neighbourhood of this point.

Example 2.1 Consider the functions $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1,x_2) = \int_0^{|x_1|} \varphi(u)\,du,$$

where function $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}$ is defined as follows:

$$\varphi(u) = \begin{cases} 1 & \text{if } u > 1, \\ 2u - \frac{1}{n+1} & \text{if } u \in \left(\frac{1}{n+1}, \frac{1}{n}\right],\ n \in \mathbb{N}, \\ 0 & \text{if } u = 0. \end{cases}$$

The first-order directional derivatives of function f at points $a_n = (\frac{1}{n}, 0)$, $n \in \mathbb{N}$, $n > 1$, in directions $\bar{d} = (1,0)$, $\hat{d} = (-1,0)$ are

$$f'(a_n;\bar{d}) = \frac{1}{n}, \qquad f'(a_n;\hat{d}) = -\frac{n+2}{n(n+1)}.$$

Hence, f is not differentiable on any neighbourhood of point x_0. For every $v = (v_1,v_2) \in S_{\mathbb{R}^2}$ and for every $y = (y_1,y_2) \in \mathbb{R}^2$, $\|y\| < 1$, $y \neq (0,0)$, it holds:

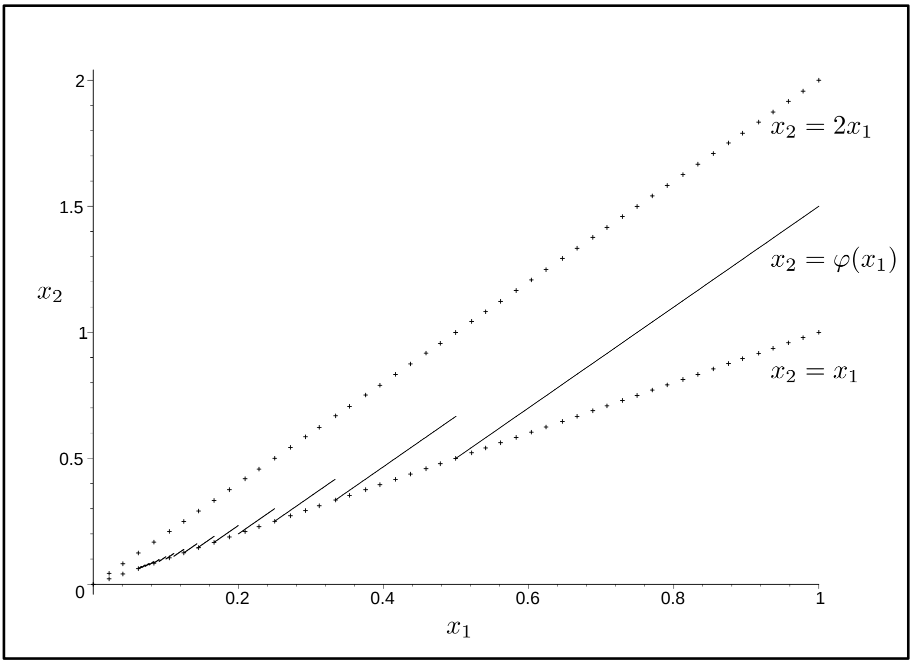

Figure 1: Graph of function φ on $[0, 1]$

- if $y_1 \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n \in \mathbb{N}$

$$\left|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\right| = \left|\lim_{t\downarrow 0} \frac{1}{t}\Big(\int_0^{y_1+tv_1} \varphi(u)\,du - \int_0^{y_1} \varphi(u)\,du\Big)\right|$$
$$= \left|\lim_{t\downarrow 0} \frac{1}{t}\Big[u^2 - \frac{u}{n+1}\Big]_{y_1}^{y_1+tv_1}\right| = |v_1|\Big(2y_1 - \frac{1}{n+1}\Big);$$

- if $y_1 \in \left(-\frac{1}{n}, -\frac{1}{n+1}\right)$, $n \in \mathbb{N}$

$$\left|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\right| = \left|\lim_{t\downarrow 0} \frac{1}{t}\Big(\int_0^{|y_1+tv_1|} \varphi(u)\,du - \int_0^{|y_1|} \varphi(u)\,du\Big)\right|$$
$$= \left|\lim_{t\downarrow 0} \frac{1}{t}\Big[u^2 - \frac{u}{n+1}\Big]_{-y_1}^{-y_1-tv_1}\right| = |v_1|\Big(2|y_1| - \frac{1}{n+1}\Big);$$

- if $y_1 = \frac{1}{n}$, $n \in \mathbb{N}$, $n > 1$, $v_1 \geq 0$

$$\left|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\right| = \left|\lim_{t\downarrow 0} \frac{1}{t}\Big(\int_0^{y_1+tv_1} \varphi(u)\,du - \int_0^{y_1} \varphi(u)\,du\Big)\right|$$
$$= \left|\lim_{t\downarrow 0} \frac{1}{t}\Big[u^2 - \frac{u}{n}\Big]_{y_1}^{y_1+tv_1}\right| = v_1 y_1;$$

- if $y_1 = \frac{1}{n}$, $n \in \mathbb{N}$, $n > 1$, $v_1 < 0$

$$\Big|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\Big| = \Big|\lim_{t\downarrow 0} \frac{1}{t}\Big(\int_0^{y_1+tv_1} \varphi(u)\,du - \int_0^{y_1} \varphi(u)\,du\Big)\Big|$$
$$= \Big|\lim_{t\downarrow 0} \frac{1}{t}\Big[u^2 - \frac{u}{n+1}\Big]_{y_1}^{y_1+tv_1}\Big| = |v_1|\Big(2y_1 - \frac{1}{n+1}\Big);$$

- if $y_1 = -\frac{1}{n}$, $n \in \mathbb{N}$, $n > 1$, $v_1 > 0$

$$\Big|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\Big| = \Big|\lim_{t\downarrow 0} \frac{1}{t}\Big(\int_0^{|y_1+tv_1|} \varphi(u)\,du - \int_0^{|y_1|} \varphi(u)\,du\Big)\Big|$$
$$= \Big|\lim_{t\downarrow 0} \frac{1}{t}\Big[u^2 - \frac{u}{n+1}\Big]_{-y_1}^{-y_1-tv_1}\Big| = |v_1|\Big(2|y_1| - \frac{1}{n+1}\Big);$$

- if $y_1 = -\frac{1}{n}$, $n \in \mathbb{N}$, $n > 1$, $v_1 \leq 0$

$$\Big|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\Big| = \Big|\lim_{t\downarrow 0} \frac{1}{t}\Big(\int_0^{|y_1+tv_1|} \varphi(u)\,du - \int_0^{|y_1|} \varphi(u)\,du\Big)\Big|$$
$$= \Big|\lim_{t\downarrow 0} \frac{1}{t}\Big[u^2 - \frac{u}{n}\Big]_{-y_1}^{-y_1-tv_1}\Big| = v_1 y_1;$$

- if $y_1 = 0$

$\Big|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\Big| = 0$ because

$$0 \leq |\lim_{t\to 0} \frac{f(y+tv)-f(y)}{t}| = |\lim_{t\to 0} \frac{f(tv)}{t}| \leq |\lim_{t\to 0} \frac{t^2 v_1^2}{t}| = 0$$

and it also implies $f'(x_0) = (0,0)$.

Overall

$$\Big|\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}\Big| = \begin{cases} |v_1|\Big(2|y_1| - \dfrac{1}{n+1}\Big) & \text{if } |y_1| \in \big(\frac{1}{n+1}, \frac{1}{n}\big), \\ v_1 y_1 & \text{if } |y_1| = \frac{1}{n},\ v_1 y_1 \geq 0, \\ |v_1|\Big(2|y_1| - \dfrac{1}{n+1}\Big) & \text{if } |y_1| = \frac{1}{n},\ v_1 y_1 < 0, \\ 0 & \text{if } y_1 = 0. \end{cases}$$

The function f is ℓ-stable at x_0 because:

$$|f^\ell(x_0; v) - f^\ell(y; v)| = |\liminf_{t\downarrow 0} \frac{f(y+tv)-f(y)}{t}| \leq 2\|y\|,$$
$$\forall y \in \mathbb{R}^2,\ \|y\| < 1,\ \forall v \in S_{\mathbb{R}^2}.$$

There are two approaches how to generalize the concept of ℓ-stability for vector functions. The first one introduced in [2] is stated in Definition 2.3. The second one was introduced in [6] and since their equivalence was shown in [5], I mention it in Theorem 2.1 as a characterization of ℓ-stability.

In the definition of ℓ-stable at a point vector function, this type of lower directional derivative is needed:

$$f_\xi^\ell(x;h) = \liminf_{t\downarrow 0} \frac{\langle \xi, f(x+th) - f(x)\rangle}{t}$$

for $f\colon \mathbb{R}^N \to \mathbb{R}^M$, $x \in \mathbb{R}^N$, $h \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^M$.

Definition 2.2 For arbitrary cone $C \subseteq \mathbb{R}^N$, we define *a positive polar cone* C^* and a set Γ_C:

$$C^* := \{\xi \in \mathbb{R}^N; \langle \xi, y\rangle \geq 0,\ y \in C\}, \qquad \Gamma_C := C^* \cap S_{\mathbb{R}^N}.$$

Definition 2.3 Let $f\colon \mathbb{R}^N \to \mathbb{R}^M$ be a function and $C \subset \mathbb{R}^M$ be a closed, convex and pointed cone with non-empty interior. We say that f is *ℓ-stable at $x_0 \in \mathbb{R}^N$ with respect to C* if there is a neighbourhood U of x_0 and a constant $K > 0$ such that

$$|f_\xi^\ell(y;h) - f_\xi^\ell(x_0;h)| \leq K\|y - x_0\|, \quad \forall y \in U, \forall h \in S_{\mathbb{R}^N}, \forall \xi \in \Gamma_C.$$

In [4], it was proved that if any function is ℓ-stable at a point with respect to some closed, convex and pointed cone, it must be ℓ-stable at this point with respect to arbitrary closed, convex and pointed cone. Therefore, we talk in the following text only about ℓ-stability at a point.

Theorem 2.1 *The function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ is* ℓ-stable at $x_0 \in \mathbb{R}^N$ *if and only if for any $\xi \in \mathbb{R}^M$ the scalar function*

$$f_\xi(\cdot) = \langle \xi, f(\cdot)\rangle$$

is ℓ-stable at x_0.

The next theorems provide characterization of ℓ-stability.

Theorem 2.2 [6, Theorem 3.3] *The function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ is ℓ-stable at $x_0 \in \mathbb{R}^N$ if and only if there exist a neighbourhood U of x_0 and a constant $K > 0$ such that it holds that*

$$\left|f_\xi^\ell(x;h) - f_\xi^\ell(x_0;h)\right| \leq K\,\|\xi\|\,\|x - x_0\|\,, \quad \forall x \in U,\ \forall h \in S_{\mathbb{R}^N},\ \forall \xi \in \mathbb{R}^M.$$

Theorem 2.3 [6, Theorem 3.4] *The function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ is ℓ-stable at $x_0 \in \mathbb{R}^N$ if and only if the Fréchet derivative $f'(x_0)$ exists, there exists a neighbourhood U of x_0 such that f is Lipschitz on U, and there is a $K > 0$ such that it holds that*

$$\|f'(x) - f'(x_0)\| \leq K\,\|x - x_0\|\,, \qquad a.e.\ x \in U.$$

At the end of this section, I mention other important properties of ℓ-stable at a point functions.

Definition 2.4 We say that a function $f: \mathbb{R}^N \to \mathbb{R}^M$ is *strictly differentiable at* $x \in \mathbb{R}^N$ if there exists a continuous linear operator $A: \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\lim_{y \to x, t \downarrow 0} \frac{f(y+th) - f(y)}{t} = Ah, \qquad \forall h \in S_{\mathbb{R}^N},$$

and this limit is uniform for $h \in S_{\mathbb{R}^N}$.

Theorem 2.4 [2, Proposition 2.2] *Let a function* $f: \mathbb{R}^N \to \mathbb{R}^M$ *be* ℓ*-stable at* $x_0 \in \mathbb{R}^N$*. Then* f *is strictly differentiable at* x_0*.*

Theorem 2.4 implies that every function which is ℓ-stable at some point is continuous near this point and Fréchet differentiable at this point.

Theorem 2.5 [4, Proposition 1] *Let* $f = (f_1, f_2, \ldots, f_M): \mathbb{R}^N \to \mathbb{R}^M$ *be a function and* $x_0 \in \mathbb{R}^N$*. Then* f *is* ℓ*-stable at* x_0 *if and only if the function* f_i *is* ℓ*-stable at* x_0 *for every* $i \in \{1, 2, \ldots, M\}$*.*

Theorem 2.6 [4, Theorem 1] *Let a function* $f: \mathbb{R}^M \to \mathbb{R}^N$ *be* ℓ*-stable at* $x_0 \in \mathbb{R}^M$ *and let a function* $g: \mathbb{R}^N \to \mathbb{R}^P$ *be* ℓ*-stable at* $y_0 = f(x_0) \in \mathbb{R}^N$*. Then the composition* $g \circ f$ *is* ℓ*-stable at* x_0*.*

3 Vector optimization results

In this section, I consider constrained optimization problem (P). Firstly I recall fundamental definitions of vector optimization and second-order directional derivatives which are used in second-order optimality conditions.

Definition 3.1 Let us consider the problem (P) and define

a) *a set of feasible points* Φ:

$$\Phi = \{x \in \mathbb{R}^N;\ g(x) \in -K\};$$

b) a cone $K(x)$, $x \in -K$:

$$K(x) = \{\gamma(z + x); \gamma \geq 0,\ z \in K\}.$$

Now we introduce two types of minimizers for problem (P).

Definition 3.2 A feasible point x_0 is said

a) *a local weakly efficient point for problem (P)* if there exists a neighbourhood U of x_0 such that

$$(f(U \cap \Phi) - f(x_0)) \cap (-\operatorname{int} C) = \emptyset.$$

b) *an isolated local minimizer of second-order for problem (P)*, if there exist a neighbourhood U of x_0 and a constant $A > 0$ such that

$$\sup_{\xi \in \Gamma_C} (\langle \xi, f(x) - f(x_0) \rangle) \geq A\|x - x_0\|^2, \quad \forall x \in U \cap \Phi.$$

Definition 3.3 Let a function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ be Fréchet differentiable at point $x \in \mathbb{R}^N$. *The second-order Dini directional derivative $d_2 f(x; h)$ of f at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$* is defined as follows:

$$d_2 f(x;h) = \Big\{ y \in \mathbb{R}^M;\ \exists \{t_n\}_{n=1}^{+\infty} \subset \mathbb{R}^+,\ \lim_{n\to+\infty} t_n = 0, \quad y = \lim_{n\to+\infty} \frac{f(x + t_n h) - f(x) - t_n f'(x)h}{t_n^2/2} \Big\}.$$

*The second-order Hadamard directional derivative $D_2 f(x; h)$ of f at $x \in \mathbb{R}^N$ in the direction $h \in \mathbb{R}^N$*is defined as follows:

$$D_2 f(x;h) = \Big\{ y \in \mathbb{R}^M;\ \exists \{t_n\}_{n=1}^{+\infty} \subset \mathbb{R}^+,\ \exists \{h_n\}_{n=1}^{+\infty} \subset \mathbb{R}^N,\ \lim_{n\to+\infty} t_n = 0, \quad \lim_{n\to+\infty} h_n = h,\ y = \lim_{n\to+\infty} \frac{f(x + t_n h_n) - f(x) - t_n f'(x)h}{t_n^2/2} \Big\}.$$

Problem (P) was deeply studied for at least continuously differentiable functions. Khanh and Tuan achieved following results using concept of calm at a point function.

Definition 3.4 A function $f\colon \mathbb{R}^N \to \mathbb{R}^M$ is called *calm at $x_0 \in \mathbb{R}^N$* if there is a neighbourhood U of x_0 and a constant $K > 0$ such that

$$\|f(x) - f(x_0)\| \leq K\|x - x_0\|, \quad \forall x \in U.$$

Theorem 3.1 [10, Theorem 4.1] *Let functions $f\colon \mathbb{R}^N \to \mathbb{R}^M$ and $g\colon \mathbb{R}^N \to \mathbb{R}^P$ be continuously differentiable at $x_0 \in \mathbb{R}^N$. If x_0 is a local weakly efficient point of problem (P), then*

(i) *there exists $(c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}^M}, 0_{\mathbb{R}^P})\}$ such that*

$$c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0_{\mathbb{R}^N}; \tag{3.1}$$

(ii) *for $u \in \mathbb{R}^N$ if $(f,g)'(x_0)u \in -(C \times K(g(x_0)) \setminus \text{int } (C \times K(g(x_0))))$, then for every $(y_0, z_0) \in D_2(f,g)(x_0; u)$ there exists $(c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}^M}, 0_{\mathbb{R}^P})\}$ such that (3.1) is true and*

$$\langle c^*, y_0 \rangle + \langle k^*, z_0 \rangle \geq 0.$$

Theorem 3.2 [10, Theorem 4.2] *Let functions $f\colon \mathbb{R}^N \to \mathbb{R}^M$ and $g\colon \mathbb{R}^N \to \mathbb{R}^P$ be continuously Fréchet differentiable around $x_0 \in \mathbb{R}^N$ with f' and g' being calm at x_0 which is feasible point of problem (P). Then, each of the following conditions is sufficient for x_0 to be an isolated local minimizer of second-order for problem (P).*

(i) For every $u \in \mathbb{R}^N$ satisfying $\|u\| = 1$ there exists $(c^*, k^*) \in C^* \times K^*(g(x_0))$ such that

$$\langle c^*, f'(x_0)u\rangle + \langle k^*, g'(x_0)u\rangle > 0.$$

(ii) For every $u \in \mathbb{R}^N$ satisfying $\|u\| = 1$, one has

a) $(f'(x_0)u, g'(x_0)u) \in -(C \times K(g(x_0)) \setminus \operatorname{int}(C \times K(g(x_0))))$,

b) for every $(y_0, z_0) \in d_2(f, g)(x_0; u)$ there exists $(c^*, k^*) \in C^* \times K^*(g(x_0))$ such that (3.1) is true and

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle > 0. \tag{3.2}$$

Theorems 3.1 and 3.2 was strengthened for strictly differentiable and ℓ-stable functions, respectively, by Bednařík and Pastor.

Theorem 3.3 [3, Theorem 3.1] *Let functions $f: \mathbb{R}^N \to \mathbb{R}^M$ and $g: \mathbb{R}^N \to \mathbb{R}^P$ be strictly differentiable at $x_0 \in \mathbb{R}^N$. If x_0 is a local weakly efficient point of problem (P), then*

(i) there exists $(c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}^M}, 0_{\mathbb{R}^P})\}$ such that

$$c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0_{\mathbb{R}^N}; \tag{3.3}$$

(ii) for $u \in \mathbb{R}^N$ if $(f, g)'(x_0)u \in -(C \times K(g(x_0)) \setminus \operatorname{int}(C \times K(g(x_0))))$, then for every $(y_0, z_0) \in D_2(f, g)(x_0; u)$ there exists $(c^*, k^*) \in C^* \times K^*(g(x_0)) \setminus \{(0_{\mathbb{R}^M}, 0_{\mathbb{R}^P})\}$ such that (3.3) is true and

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle \geq 0.$$

Theorem 3.4 [3, Theorem 4.1], [6, proof of Thm 5.1] *Let functions $f: \mathbb{R}^N \to \mathbb{R}^M$ and $g: \mathbb{R}^N \to \mathbb{R}^P$ be ℓ-stable at feasible point $x_0 \in \mathbb{R}^N$. We suppose that for every $u \in S_{\mathbb{R}^N}$ one of the following two conditions is satisfied.*

(i) There exists $(c^*, k^*) \in C^* \times K^*(g(x_0))$ such that

$$\langle c^*, f'(x_0)u\rangle + \langle k^*, g'(x_0)u\rangle > 0.$$

(ii)

a) $(f'(x_0)u, g'(x_0)u) \in -(C \times K(g(x_0)) \setminus \operatorname{int}(C \times K(g(x_0))))$,

b) for every $(y_0, z_0) \in d_2(f, g)(x_0; u)$ there exists $(c^*, k^*) \in C^* \times K^*(g(x_0))$ such that

$$c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0_{\mathbb{R}^N}, \tag{3.4}$$
$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle > 0. \tag{3.5}$$

Then x_0 is an isolated local minimizer of second-order for problem (P).

In [6, Theorem 5.1], the condition (3.4) from Theorem 3.4 is substituted by

$$\langle c^*, f'(x_0)u\rangle + \langle k^*, g'(x_0)u\rangle = 0. \tag{3.6}$$

In [5], the incorrectness of using condition (3.6) was showed on following example.

Example 3.1 Let us consider the problem (P) with the functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, and $g\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = (x_1 + x_2^2, x_1^2), \qquad g(x_1, x_2) = x_1 x_2,$$

$C = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2; x_1 \geq 0, x_2 \geq 0\}$, $K = [0, +\infty)$.

It can be showed that these functions fulfill at point $x_0 = (0,0)$ the assumptions of Theorem 3.4 where (3.4) is replaced by (3.6) but x_0 is not an isolated local minimizer of second-order for problem (P). The condition (i) is satisfied for $u = (u_1, u_2) \in S_{\mathbb{R}^2}$, $u_1 > 0$, choosing $c^* = (1, 0) \in C^*$ and arbitrary $k^* \in K^*(g(x_0))$:

$$\langle c^*, f'(x_0)u\rangle + \langle k^*, g'(x_0)u\rangle = \langle (1,0), (u_1, 0)\rangle + 0 = u_1 > 0.$$

The changed condition (ii) is satisfied for $u = (u_1, u_2) \in S_{\mathbb{R}^2}$, $u_1 = 0$, choosing $c^* = (1, 0)$, $k^* = 0$ and for $u_1 < 0$, choosing $c^* = (0, 1)$, $k^* = 0$:

$$(f'(x_0)u, g'(x_0)u) = (u_1, 0, 0) \in -(C \times K(g(x_0)) \setminus \operatorname{int}(C \times K(g(x_0)))),$$

$$\langle c^*, f'(x_0)u\rangle + \langle k^*, g'(x_0)u\rangle = \begin{cases} \langle (1,0), (0,0)\rangle + 0 = 0, & \text{if } u_1 = 0, \\ \langle (0,1), (u_1, 0)\rangle + 0 = 0, & \text{if } u_1 < 0, \end{cases}$$

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle = \begin{cases} \langle (1,0), (2u_2^2, 0)\rangle + 0 = 2u_2^2 > 0, & \text{if } u_1 = 0, \\ \langle (0,1), (2u_2^2, 2u_1^2)\rangle + 0 = 2u_1^2 > 0, & \text{if } u_1 < 0. \end{cases}$$

However, x_0 is not an isolated local minimizer of second-order, since the sequence of feasible points $\left\{\left(-\frac{1}{k}, \sqrt{\frac{1}{k}}\right)\right\}_{k=1}^{+\infty}$ converges to x_0 for $k \to +\infty$, but for every $A > 0$, it can be found $k_0 \in \mathbb{N}$ such that it holds for every $k \in \mathbb{N}$, $k \geq k_0$:

$$\sup_{\xi \in \Gamma} \left\langle \xi, f\left(-\frac{1}{k}, \sqrt{\frac{1}{k}}\right)\right\rangle = \left\langle (0,1), \left(0, \frac{1}{k^2}\right)\right\rangle = \frac{1}{k^2} < A\left\|\left(-\frac{1}{k}, \sqrt{\frac{1}{k}}\right)\right\|^2 = \frac{A}{k^2}(1+k).$$

Thus x_0 cannot be an isolated local minimizer of second-order for problem (P).

4 Conclusion

This paper informed about ℓ-stable at a point vector functions, their characterizations and properties and about their applications in second-order optimality conditions for constrained vector optimization problem. I tried to sum up the most important results to provide insight into these issues. The research of ℓ-stability and its application continues. Currently it is focused on ℓ-stability in infinite-dimensional normed linear spaces.

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