Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **53**, 1 (2014) 25–43

On Almost Pseudo-Z-symmetric Manifolds

Uday Chand DE[1], Prajjwal PAL[2]

[1] *Department of Pure Mathematics, University of Calcutta*
35, Ballygaunge Circular Road, Kolkata 700019,
West Bengal, India
e-mail: uc_de@yahoo.com

[2] *Chakdaha Co-operative Colony Vidyayatan (H.S)*
P.O.- Chakdaha, Dist- Nadia, West Bengal, India
e-mail: prajjwalpal@yahoo.in

(Received May 10, 2013)

Abstract

The object of the present paper is to study almost pseudo-Z-symmetric manifolds. Some geometric properties have been studied. Next we consider conformally flat almost pseudo-Z-symmetric manifolds. We obtain a sufficient condition for an almost pseudo-Z-symmetric manifold to be a quasi Einstein manifold. Also we prove that a totally umbilical hypersurface of a conformally flat $A(PZS)_n$ $(n > 3)$ is a manifold of quasi constant curvature. Finally, we give an example to verify the result already obtained in Section 5.

Key words: pseudo symmetric manifolds, pseudo Ricci symmetric manifolds, almost pseudo Ricci symmetric manifolds, almost pseudo-Z-symmetric manifolds, conformally flat almost pseudo-Z-symmetric manifolds

2010 Mathematics Subject Classification: 53C25

1 Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [3], who, in particular, obtained a classification of those spaces. Let (M^n, g), $(n = \dim M)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g, and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [31]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to different extent such as conformally symmetric manifolds by Chaki and Gupta [4], recurrent manifolds introduced by Walker [41], conformally recurrent manifolds by Adati and Miyazawa [1], pseudo symmetric manifolds by Chaki [5], weakly symmetric manifolds by Tamássy and Binh [39] etc.

H. Takeno and M. Ikeda [38] considered geodesic mappings from spherically symmetric four dimensional Riemannian spaces and N. S. Sinyukov [36] proved that the symmetric and recurrent Riemannian spaces, with nonconstant curvature do not admit non-trivial geodesic mappings. There are analogous results for more general recurrent manifolds published J. Mikeš in [27], and, moreover, general results obtained in ([21], [30], [28], [29]).

A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is said to be a pseudo symmetric manifold [5] if its curvature tensor satisfies the condition

$$(\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho,$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X),$$

for all X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. The class of pseudo symmetric manifolds arose during the study of conformally flat space of class one [37]. The notion of weakly symmetric manifold was introduced by Tamássy and Binh [39]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly symmetric if the curvature tensor R of type (0,4) satisfies the condition

$$(\nabla_X R)(Y,Z,W,V) = A(X)R(Y,Z,W,V) + B(Y)R(X,Z,W,V) \\ + C(Z)R(Y,X,W,V) + D(W)R(Y,Z,X,V) + E(V)R(Y,Z,W,X), \quad (1.1)$$

where ∇ denotes the Levi-Civita connection on (M^n, g), and A, B, C, D and E are 1-forms respectively which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. It was proved in [11] that the 1-forms must be related as follows $B(X) = C(X)$ and $D(X) = E(X)$.

That is, the weakly symmetric manifold is characterized by the condition

$$(\nabla_X R)(Y,Z,W,V) = A(X)R(Y,Z,W,V) + B(Y)R(X,Z,W,V) \\ + B(Z)R(Y,X,W,V) + D(W)R(Y,Z,X,V) + D(V)R(Y,Z,W,X), \quad (1.2)$$

where A, B, D are 1-forms (not simultaneously zero). The 1-forms A, B and D are called the associated 1-forms. If in (1.2) the 1-form A is replaced by $2A$;

B and D are replaced by A, then the manifold (M^n, g) reduces to a pseudo symmetric manifold in the sense of Chaki [5]. Again if $A = B = D = 0$, the manifold reduces to a symmetric manifold in the sense of Cartan. The existence of a $(WS)_n$ was proved by Prvanović [34] and a concrete example is given by De and Bandyopadhyay ([11],[12]).

Weakly symmetric manifolds have been studied by several authors ([2], [10], [17], [22], [32], [33], [43]). This justifies the name weakly symmetric manifold defined by (1.1). In 1993 Tamássy and Binh [40] introduced the notion of weakly Ricci symmetric manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(Y, X), \tag{1.3}$$

where A, B, C are three non-zero 1-forms, and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such an n-dimensional manifold is denoted by $(WRS)_n$. If in (1.3) the 1-form A is replaced by $2A$; B and C are replaced by A, then the manifold is called a pseudo Ricci symmetric manifold introduced by Chaki [6]. This implies that pseudo Ricci symmetric manifold is a particular case of a weakly Ricci symmetric manifold defined by (1.3).

The notion of an almost pseudo Ricci symmetric manifold was introduced by Chaki and Kawaguchi [7]. It was a generalization of the notion of pseudo Ricci symmetric manifold and was defined as follows:

A non-flat Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), \tag{1.4}$$

where A and B are two 1-forms and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In such a case A and B are called the associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. If $B = A$, then the (1.4) takes the following form:

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),$$

which is called a pseudo Ricci symmetric manifold introduced by Chaki [6]. Let

$$g(X, P) = A(X) \text{ and } g(X, Q) = B(X), \quad \text{for all } X. \tag{1.5}$$

Then P, Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B, respectively.

In a recent paper [24] Mantica and Molinari introduced weakly-Z-symmetric manifolds which is denoted by $(WZS)_n$. It was a generalization of the notion of weakly Ricci symmetric manifolds, pseudo Ricci symmetric manifolds, pseudo projective Ricci symmetric manifolds. A (0,2) symmetric tensor is a generalized Z tensor if

$$Z(X, Y) = S(X, Y) + \phi g(X, Y), \tag{1.6}$$

where ϕ is an arbitrary scalar function. The scalar Z is obtained by contracting (1.6) over X and Y as follows:

$$Z = r + n\phi. \tag{1.7}$$

A manifold is called almost pseudo-Z-symmetric and denoted by $A(PZS)_n$, if the generalized Z tensor is non-zero and satisfies the condition (1.4), that is,

$$(\nabla_X Z)(Y,W) = [A(X) + B(X)]Z(Y,W) + A(Y)Z(X,W) + A(W)Z(X,Y), \tag{1.8}$$

where Z is the generalized Z tensor. The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}r$, where r is the scalar curvature. Hereafter we refer to the generalized Z tensor simply as the Z tensor.

On the otherhand, quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition:

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions and η is a non-zero 1-form such that

$$g(X,\xi) = \eta(X),$$

for all vector fields X. The quasi Einstein manifold is denoted by $(QE)_n$.

A. Gray [20] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. A Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfy the condition

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{1.9}$$

Again a Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfy the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{1.10}$$

In a recent paper De and Gazi [13] studied almost pseudo symmetric manifolds. In subsequent papers ([14], [15]) De and Gazi studied almost pseudo conformally symmetric manifolds and conformally flat almost pseudo Ricci symmetric manifolds.

If $B = A$ in (1.8) then the manifold is called pseudo-Z-symmetric manifold and denoted by $(PZS)_n$. Pseudo-Z-symmetric and recurrent Z forms on Riemannian manifolds have been studied in [25] and [26] respectively. It may be mentioned that any $(PZS)_n$ is a particular case of an $A(PZS)_n$, but a $(WZS)_n$ is not a particular case of an $A(PZS)_n$. So it is interesting to study $A(PZS)_n$. In a recent paper [15] De and Gazi give two examples of $A(PRS)_n$. Among these two examples one is conformally flat $A(PRS)_4$ and another one is non-conformally flat $A(PRS)_n$. Also in a recent paper De, Őzgűr and De [18] studied conformally flat almost pseudo Ricci symmetric space-time. Motivated by these works we study $A(PZS)_n$ $(n > 2)$ in the present paper.

We also have a very useful lemma as follows:

Walker's Lemma. [41] *If a_{ij}, b_i are numbers satisfying $a_{ij} = a_{ji}$, and*

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0,$$

for $i, j, k = 1, 2, \ldots, n$, then either all $a_{ij} = 0$ or, all b_i are zero.

The paper is organized as follows: After preliminaries in Section 3, we study $A(PZS)_n$ $(n > 2)$ with cyclic parallel Z-tensor. In Section 4, we consider $A(PZS)_n$ with Codazzi type of Z-tensor. We prove that in an $A(PZS)_n$ if the Z-tensor is of Codazzi type, then the manifold reduces to a quasi Einstein manifold. Section 5 is devoted to study conformally flat $A(PZS)_n$ $(n > 3)$ and it is shown that such a manifold is a quasi Einstein manifold. Next we obtain a sufficient condition for an $A(PZS)_n$ to be a quasi Einstein manifold. In Section 7, we show that a totally umbilical hypersurface of a conformally flat $A(PZS)_n$ $(n > 3)$ is a manifold of quasi constant curvature. Finally, we give an example of an $A(PZS)_4$ to verify the result already obtained in Section 5.

2 Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively. L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(LX, Y) = S(X, Y), \tag{2.1}$$

for any vector fields X, Y. Let $\bar{A}$ and $\bar{B}$ are two 1-forms defined by

$$A(LX) = \bar{A}(X), \qquad B(LX) = \bar{B}(X). \tag{2.2}$$

Then $\bar{A}$ and $\bar{B}$ are called auxiliary 1-forms corresponding to the 1-forms A and B respectively. We have from (1.6)

$$Z(X, Y) = Z(Y, X), \tag{2.3}$$

and

$$Z(Y, Q) = S(Y, Q) + \phi g(Y, Q),$$

or,

$$Z(Y, Q) = \bar{B}(Y) + \phi B(Y). \tag{2.4}$$

Also we obtain from (1.8)

$$(\nabla_X Z)(Y, W) - (\nabla_W Z)(X, Y) = B(X)Z(Y, W) - B(W)Z(X, Y). \tag{2.5}$$

Using (1.6) in (2.5) we get

$$\begin{aligned}(\nabla_X S)(Y, W) + (X\phi)g(Y, W) - (\nabla_W S)(X, Y) - (W\phi)g(X, Y)\\ = B(X)Z(Y, W) - B(W)Z(Y, X).\end{aligned} \tag{2.6}$$

Now contracting (2.6) over Y, W and using (1.7) and (2.4) we get

$$dr(X) = \{2r + 2(n-1)\phi\}B(X) - 2\bar{B}(X) - 2(n-1)(X\phi). \tag{2.7}$$

A conformally flat Riemannian manifold (M^n, g) $(n > 3)$ is said to be quasi-constant curvature [8] if its curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\begin{aligned}\tilde{R}(X,Y,U,W) = p[g(Y,U)g(X,W) - g(X,U)g(Y,W)]\\ + q[g(X,W)H(Y)H(U) + g(Y,U)H(X)H(W)\\ - g(X,U)H(Y)H(W) - g(Y,W)H(X)H(U)],\end{aligned} \tag{2.8}$$

where $\tilde{R}(X,Y,U,W) = g(R(X,Y)U,W)$, and R is the curvature tensor of type (1,3), p, q are scalar functions of which $q \neq 0$ and H is a non-zero 1-form defined by $g(X,\mu) = H(X)$ for all X, μ being a unit vector field.

In such a case p and q are called associated scalars, H is called the associated 1-form and μ is called the generator of the manifold.

In 1956, S. S. Chern [9] studied a type of Riemannian manifold whose curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\tilde{R}(X,Y,U,W) = F(Y,U)F(X,W) - F(X,U)F(Y,W), \tag{2.9}$$

where F is a symmetric tensor of type (0,2). Such an n-dimensional manifold was called a special manifold with the associated symmetric tensor F and was denoted by $\psi(F)_n$. Such a manifold is important for the following reasons:

Firstly, for possessing some remarkable properties relating to curvature and characteristic classes and secondly, for containing a manifold of quasi-constant curvature as a subclass.

3 $A(PZS)_n$ $(n > 2)$ with cyclic parallel Z tensor

In (1.8) if we replace Y, W by X we get

$$(\nabla_X Z)(X,X) = [A(X) + B(X)]Z(X,X) + A(X)Z(X,X) + A(X)Z(X,X).$$

or,

$$(\nabla_X Z)(X,X) = [3A(X) + B(X)]Z(X,X). \tag{3.1}$$

By hypothesis the Z tensor is non-zero, hence from (3.1) it follows that

$$(\nabla_X Z)(X,X) = 0 \quad \text{if and only if} \quad 3A(X) + B(X) = 0.$$

Hence we have the following theorem:

Theorem 3.1 *In an $A(PZS)_n$ the Z tensor is covariantly constant in the direction of X if and only if $3A(X) + B(X) = 0$.*

Again interchanging X, Y, W in (1.8) and then adding we get

$$(\nabla_X Z)(Y,W) + (\nabla_Y Z)(X,W) + (\nabla_W Z)(X,Y)$$
$$= F(X)Z(Y,W) + F(Y)Z(X,W) + F(W)Z(X,Y), \tag{3.2}$$

where $F(X) = 3A(X) + B(X)$. Now if the Z tensor of the manifold be cyclic parallel, then we have

$$(\nabla_X Z)(Y,W) + (\nabla_Y Z)(X,W) + (\nabla_W Z)(X,Y) = 0. \tag{3.3}$$

From (3.2) we get

$$F(X)Z(Y,W) + F(Y)Z(X,W) + F(W)Z(X,Y) = 0. \tag{3.4}$$

Then by Walker's lemma we can see that either, $F(X) = 0$ or, $Z(X,Y) = 0$ for all X, Y. But since $Z(X,Y) \neq 0$, we have $F(X) = 0$ for all X, which implies that

$$3A(X) + B(X) = 0. \tag{3.5}$$

Conversely, if $3A(X) + B(X) = 0$, then from(3.2) we obtain

$$(\nabla_X Z)(Y,W) + (\nabla_Y Z)(X,W) + (\nabla_W Z)(X,Y) = 0$$

which implies that the Z tensor is cyclic parallel. Thus we can state the following theorem:

Theorem 3.2 *In an $A(PZS)_n$ $(n > 2)$ the Z-tensor is cyclic parallel if and only if the associated 1-forms A and B satisfy the relation (3.5).*

Let the Z tensor of the manifold be cyclic parallel. Then the associated 1-forms A and B satisfy the relation (3.5) from which we get

$$A(X) = -\frac{1}{3}B(X). \tag{3.6}$$

Hence if the 1-form B is closed, then from (3.6) we obtain that A is also closed and conversely. Thus we have the following:

Corollary 3.1 *In an $A(PZS)_n$ $(n > 2)$ if the Z tensor is cyclic parallel, then the 1-form A is closed if and only if the 1-form B is also closed.*

4 An $A(PZS)_n$ $(n > 2)$ with Codazzi type of Z tensor

Here we suppose that the Z-tensor in $A(PZS)_n$ is of Codazzi type. Now from (1.8) we get

$$(\nabla_X Z)(Y,W) - (\nabla_W Z)(X,Y) = B(X)Z(Y,W) - B(W)Z(X,Y). \tag{4.1}$$

Since Z is of Codazzi type we have from (4.1)

$$B(X)Z(Y,W) - B(W)Z(X,Y) = 0. \tag{4.2}$$

Putting $X = Q$ in (4.2) we get

$$B(Q)Z(Y,W) = B(W)Z(Q,Y). \tag{4.3}$$

Again putting $Y = W = e_i$ in (4.2) and taking summation over $i, 1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$B(X)Z = Z(X,Q), \tag{4.4}$$

where $Z = \Sigma_{i=1}^{n} Z(e_i, e_i)$. Using (4.4) in (4.3) we get

$$Z(Y,W) = \frac{ZB(Y)B(W)}{B(Q)}. \tag{4.5}$$

Now using (1.6) in (4.5) we get

$$S(Y,W) = -\phi g(Y,W) + tB(Y)B(W),$$

where $t = \frac{Z}{B(Q)}$. Thus we have

$$S(X,Y) = ag(X,Y) + bB(X)B(Y), \tag{4.6}$$

where $a = -\phi$ and $b = t = \frac{Z}{B(Q)}$. Thus the manifold is a quasi-Einstein manifold. Hence we have the following theorem:

Theorem 4.1 *If the Z tensor in $A(PZS)_n$ is of Codazzi type, then the manifold reduces to a quasi Einstein manifold.*

5 Conformally flat $A(PZS)_n$ $(n > 3)$

In this section we assume that the manifold $A(PZS)_n$ is conformally flat. Then $\operatorname{div} C = 0$ where C denotes the Weyl's conformal curvature tensor and 'div' denotes divergence. Hence we have [19]

$$(\nabla_X S)(Y,W) - (\nabla_W S)(X,Y) = \frac{1}{2(n-1)}[g(Y,W)dr(X) - g(X,Y)dr(W)]. \tag{5.1}$$

Using (2.6) and (2.7) in (5.1) we get

$$\begin{aligned}
&B(X)Z(Y,W) - B(W)Z(X,Y) - (X\phi)g(Y,W) + (W\phi)g(X,Y) \\
&= \frac{1}{2(n-1)}[B(X)g(Y,W)\{2r + 2(n-1)\phi\} - 2(n-1)(X\phi)g(Y,W) \\
&\quad - 2\bar{B}(X)g(Y,W) - B(W)g(X,Y)\{2r + 2(n-1)\phi\} \\
&\quad + 2(n-1)(W\phi)g(X,Y) + 2\bar{B}(W)g(X,Y)].
\end{aligned} \tag{5.2}$$

Now putting $Y = Q$ in (5.2) we get

$$B(X)\bar{B}(W) = \bar{B}(X)B(W). \tag{5.3}$$

Again putting $X = Q$ in (5.3) we get

$$B(Q)\bar{B}(W) = \bar{B}(Q)B(W).$$

or,

$$\bar{B}(W) = \frac{\bar{B}(Q)}{B(Q)}B(W).$$

or,

$$\bar{B}(W) = sB(W), \tag{5.4}$$

where

$$s = \frac{\bar{B}(Q)}{B(Q)}, \tag{5.5}$$

and s is a scalar. Now using (5.4) in (2.7) we get

$$dr(X) = \{2r + 2(n-1)\phi\}B(X) - 2sB(X) - 2(n-1)(X\phi).$$

or,

$$dr(X) = 2\{r - s + (n-1)\phi\}B(X) - 2(n-1)(X\phi). \tag{5.6}$$

Since $B \neq 0$, putting $X = Q$ in (5.2) and using (5.4) we get

$$\begin{aligned} &B(Q)Z(Y,W) - B(W)Z(Q,Y) \\ &= \frac{1}{2(n-1)}[\{2r + 2(n-1)\phi\}B(Q)g(Y,W) - 2sB(Q)g(Y,W) \\ &\quad - \{2r + 2(n-1)\phi\}B(Y)B(W) + 2sB(W)B(Y)]. \end{aligned} \tag{5.7}$$

Using (1.6), (5.4) in (5.7) we get

$$\begin{aligned} &B(Q)S(Y,W) + \phi B(Q)g(Y,W) - sB(W)B(Y) - \phi B(Y)B(W) \\ &= \frac{1}{n-1}[\{r + (n-1)\phi - s\}B(Q)g(Y,W) - \{r + (n-1)\phi - s\}B(Y)B(W)]. \end{aligned}$$

or,

$$\begin{aligned} B(Q)S(Y,W) &= \left\{\frac{r + (n-1)\phi - s}{n-1} - \phi\right\}B(Q)g(Y,W) \\ &+ \left\{(s+\phi) - \frac{r + (n-1)\phi - s}{n-1}\right\}B(Y)B(W). \end{aligned}$$

or,

$$S(Y,W) = \left(\frac{r-s}{n-1}\right)g(Y,W) + \left(\frac{ns-r}{n-1}\right)\frac{B(Y)B(W)}{B(Q)}.$$

or,

$$S(Y,W) = ag(Y,W) + bT(Y)T(W), \tag{5.8}$$

where $a = \frac{r-s}{n-1}$, $b = \frac{ns-r}{n-1}$ are scalars and $T(X) = \frac{B(X)}{\sqrt{B(Q)}}$. A Riemannian manifold is said to be a quasi-Einstein manifold if its Ricci tensor is of the form (5.8). Hence we have the following theorem:

Theorem 5.1 *A confomally flat $A(PZS)_n$ $(n > 3)$ is a quasi-Einstein manifold.*

Now from (5.8) we have

$$S(Y,W) = \left(\frac{r-s}{n-1}\right) g(Y,W) + \left(\frac{ns-r}{n-1}\right) \frac{B(Y)B(W)}{B(Q)}. \tag{5.9}$$

Putting $W = Q$ in (5.9) we get

$$S(Y,Q) = sB(Y) = sg(Y,Q), \tag{5.10}$$

Thus we can state the following:

Corollary 5.1 *The vector field Q corresponding to the 1-form B is an eigen vector of the Ricci tensor S corresponding to the eigen value s.*

Let us suppose that the associated vector field Q corresponding to the 1-form B is a unit vector field. Therefore,

$$T(X) = B(X), \tag{5.11}$$

since $B(Q) = 1$. In a conformally flat Riemannian manifold the curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition [19]

$$\begin{aligned} \tilde{R}(X,Y,U,W) = \frac{1}{(n-2)}[&S(Y,U)g(X,W) \\ - S(X,U)g(Y,W) + S(X,W)g(Y,U) &- S(Y,W)g(X,U)] \\ - \frac{r}{(n-1)(n-2)}[g(Y,U)g(X,W) &- g(X,U)g(Y,W)], \end{aligned} \tag{5.12}$$

where $\tilde{R}(X,Y,U,W) = g(R(X,Y)U,W)$, R is the Riemannian curvature tensor of type (1,3), and r is the scalar curvature. Now using (5.8) and (5.11) in (5.12) we get

$$\begin{aligned} \tilde{R}(X,Y,U,W) = \frac{1}{n-2}[&ag(Y,U)g(X,W) + bB(Y)B(U)g(X,W) \\ - ag(X,U)g(Y,W) - bB(X)B(U)g(Y,W) &+ ag(X,W)g(Y,U) \\ + bB(X)B(W)g(Y,U) - ag(Y,W)g(X,U) &- bB(Y)B(W)g(X,U)] \\ - \frac{r}{(n-1)(n-2)}[g(Y,U)g(X,W) &- g(X,U)g(Y,W)]. \end{aligned}$$

or,

$$\begin{aligned} \tilde{R}(X,Y,U,W) = \left[\frac{2a}{(n-2)} - \frac{r}{(n-1)(n-2)}\right] &[g(Y,U)g(X,W) \\ - g(X,U)g(Y,W)] + \left[\frac{b}{n-2}\right] [g(X,W)B(Y)B(U) &+ g(Y,U)B(X)B(W) \\ - g(X,U)B(Y)B(W) - g(Y,W)B(X)B(U)]. \end{aligned}$$

or,

$$\begin{aligned}\tilde{R}(X,Y,U,W) &= p[g(Y,U)g(X,W) - g(X,U)g(Y,W)] \\ &+ q[g(X,W)B(Y)B(U) + g(Y,U)B(X)B(W) \\ &- g(X,U)B(Y)B(W) - g(Y,W)B(X)B(U)],\end{aligned}$$

where $p = \frac{r-2s}{(n-1)(n-2)}$, $q = \frac{ns-r}{(n-1)(n-2)}$. This implies that the manifold is of quasi-constant curvature. Thus we can state the following theorem:

Theorem 5.2 *A conformally flat $A(PZS)_n$ ($n > 3$) is a manifold of quasi-constant curvature provided the vector field metrically equivalent to the 1-form B is a unit vector field.*

Now, let us suppose that in a manifold of quasi-constant curvature

$$F(X,Y) = \sqrt{p}g(X,Y) + \frac{q}{\sqrt{p}}H(X)H(Y). \tag{5.13}$$

It is obvious that

$$F(X,Y) = F(Y,X). \tag{5.14}$$

Thus F is a symmetric tensor of type (0,2). Now (2.9) can be written as

$$\tilde{R}(X,Y,U,W) = F(Y,U)F(X,W) - F(X,U)F(Y,W).$$

Thus a manifold of quasi-constant curvature is a $\psi(F)_n$. Hence a $\psi(F)_n$ contains a manifold of quasi-constant curvature as a subclass. So we have the following:

Proposition 5.1 *A manifold of quasi-constant curvature is a $\psi(F)_n$.*

From this Proposition 5.1 and Theorem 5.2 we can conclude that

Corollary 5.2 *A conformally flat $A(PZS)_n$ ($n > 3$) is a $\psi(F)_n$.*

6 Sufficient condition for an $A(PZS)_n$ to be a quasi Einstein manifold

In an $A(PZS)_n$ the Z tensor satisfies

$$(\nabla_U Z)(X,Y) = [A(U)+B(U)]Z(X,Y)+B(X)Z(U,Y)+B(Y)Z(U,X). \tag{6.1}$$

In a Riemannian manifold a vector field P defined by $g(X,P) = A(X)$ for all vector field X is said to be a concircular vector field [35] if

$$(\nabla_X A)(Y) = \alpha g(X,Y) + \omega(X)A(Y), \tag{6.2}$$

where α is a non-zero scalar and ω is a closed 1-form. If P is a unit one then the equation (6.2) can be written as

$$(\nabla_X A)(Y) = \alpha[g(X,Y) - A(X)A(Y)]. \tag{6.3}$$

We suppose that in an $A(PZS)_n$ the vector field P is a unit concircular vector field defined by (6.3) where α is a non-zero scalar. Applying Ricci identity to (6.3) we obtain

$$A(R(X,Y)Z) = \alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)]. \tag{6.4}$$

Putting $Y = Z = e_i$ in (6.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$A(LX) = (n-1)\alpha^2 A(X),$$

where L is the Ricci operator defined by $g(LX,Y) = S(X,Y)$, which implies that

$$S(X,P) = (n-1)\alpha^2 A(X). \tag{6.5}$$

Now,

$$(\nabla_Y S)(X,P) = \nabla_Y S(X,P) - S(\nabla_Y X, P) - S(X, \nabla_Y P). \tag{6.6}$$

Applying (6.5) and (6.3) in (6.6) we get

$$(\nabla_Y S)(X,P) = (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] - S(X, \nabla_Y P). \tag{6.7}$$

Since $(\nabla_X g)(Y,P) = 0$, we have

$$(\nabla_Y A)(X) = g(X, \nabla_Y P). \tag{6.8}$$

Using (6.3) in (6.8) yields

$$\alpha[g(X,Y) - A(X)A(Y)] = g(X, \nabla_Y P)$$

which implies

$$\nabla_Y P = \alpha Y - \alpha A(Y)P = \alpha(Y - A(Y)P).$$

Hence

$$S(X, \nabla_Y P) = \alpha[S(X,Y) - A(Y)S(X,P)]. \tag{6.9}$$

Applying (6.9) in (6.7) we get

$$\begin{aligned}(\nabla_Y S)(X,P) = {} & (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] \\ & - \alpha S(X,Y) + \alpha A(Y)S(X,P). \end{aligned} \tag{6.10}$$

Again using (6.5) in (6.10) we get

$$(\nabla_Y S)(X,P) = (n-1)\alpha^3 g(X,Y) - \alpha S(X,Y). \tag{6.11}$$

Using (1.6) in (6.1) we get

$$\begin{aligned}(\nabla_U S)(X,Y) + (U\phi)g(X,Y) = {} & [A(U) + B(U)]Z(X,Y) \\ & + B(X)Z(U,Y) + B(Y)Z(U,X). \end{aligned} \tag{6.12}$$

Putting $Y = P$ and using (6.5) and (6.11) in (6.12) we get

$$\begin{aligned}&(n-1)\alpha^3 g(X,U) - \alpha S(X,U) + (U\phi)A(X)\\ &= [A(U)+B(U)][(n-1)\alpha^2 A(X) + \phi A(X)] + B(X)[(n-1)\alpha^2\\ &\quad + \phi]A(U) + B(P)[S(U,X) + \phi g(U,X)],\end{aligned}$$

which implies

$$\begin{aligned}&[\alpha + B(P)]S(U,X) = (n-1)\alpha^3 g(U,X)\\ &- [A(U)+B(U)]\{\phi + (n-1)\alpha^2\}A(X) - \phi B(P) g(U,X)\\ &- \{(n-1)\alpha^2 + \phi\}A(U)B(X) + (U\phi)A(X). \qquad (6.13)\end{aligned}$$

Putting $X = P$ in (6.13) and using (6.5) we get

$$\begin{aligned}&[\alpha + B(P)](n-1)\alpha^2 A(U) = (n-1)\alpha^3 A(U)\\ &- [A(U)+B(U)]\{\phi + (n-1)\alpha^2\}A(P) - \phi B(P)A(U)\\ &- \{(n-1)\alpha^2 + \phi\}A(U)B(P) + (U\phi)A(P). \qquad (6.14)\end{aligned}$$

From (6.14) it follows that

$$B(U) = -[2B(P)+1]A(U) - \frac{(U\phi)}{\phi + (n-1)\alpha^2}. \qquad (6.15)$$

Let us suppose

$$\alpha + B(P) \neq 0.$$

Using (6.15) in (6.13) we have

$$\begin{aligned}S(U,X) &= \frac{(n-1)\alpha^3 g(U,X)}{\alpha + B(P)} - \Bigg[A(U) - \{2B(P)+1\}A(U)\\ &+ \frac{(U\phi)}{\phi + (n-1)\alpha^2}\Bigg]\frac{\{(n-1)\alpha^2 + \phi\}A(X)}{\alpha + B(P)} - \frac{\phi B(P) g(U,X)}{\alpha + B(P)}\\ &- \frac{\{(n-1)\alpha^2 + \phi\}A(U)}{\alpha + B(P)}\left[-\{2B(P)+1\}A(X) + \frac{(U\phi)}{(n-1)\alpha^2 + \phi}\right] + \frac{(U\phi)A(X)}{\alpha + B(P)}.\end{aligned}$$

or,

$$\begin{aligned}S(U,X) &= \frac{(n-1)\alpha^3 - \phi B(P)}{\alpha + B(P)} g(U,X)\\ &+ \frac{\{(n-1)\alpha^2 + \phi\}\{4B(P)+1\}}{\alpha + B(P)} A(U)A(X) - \frac{(U\phi)A(U)}{\alpha + B(P)}.\end{aligned}$$

Now if we assume that $\phi = \textit{constant}$ then the above equation implies that the manifold under consideration is a quasi Einstein manifold. Thus we are in a position to state the following:

Theorem 6.1 *If in an $A(PZS)_n$, the basic vector field P is a unit concircular vector field, then the manifold is a quasi Einstein manifold provided*

$$\alpha + B(P) \neq 0$$

and $\phi =$ constant.

7 Totally umbilical hypersurface of a conformally flat $A(PZS)_n$ $(n > 3)$

In this section we consider a hypersurface $(\bar{M}^{n-1}, g)$ of a conformally flat $A(PZS)_n$ $(n > 3)$ and denote the curvature tensor of the hypersurface by $\bar{R}$. Then for any vector fields X, Y, U, W tangent to $\bar{M}$ we have the following equation of Gauss ([42], p. 68)

$$g(R(X,Y,U),W) = \\ = g(\bar{R}(X,Y,U),W) - g(B(X,W),B(Y,U)) + g(B(Y,W),B(X,U)), \quad (7.1)$$

where R is the curvature tensor of the $A(PZS)_n$ and B is the second fundamental form of $\bar{M}$. If

$$B(X,Y) = \tau g(X,Y) \quad (7.2)$$

for any vector fields X, Y tangent to $\bar{M}$, where τ is the mean curvature vector of $\bar{M}$, then $\bar{M}$ is said to be totally umbilical ([42], p. 67). It is known from Theorem 5.2 that in a conformally flat $A(PZS)_n$ $(n > 3)$ the curvature tensor R satisfies the following condition:

$$\begin{aligned} g(R(X,Y,U),W) = {} & p[g(Y,U)g(X,W) - g(X,U)g(Y,W)] \\ & + q[g(X,W)T(Y)T(U) + g(Y,U)T(X)T(W) \\ & - g(X,U)T(Y)T(W) - g(Y,W)T(X)T(U)], \end{aligned} \quad (7.3)$$

where $p = \frac{r-2s}{(n-1)(n-2)}$, $q = \frac{ns-r}{(n-1)(n-2)}$ and

$$T(X) = \frac{B(X)}{\sqrt{B(Q)}}, \quad B(Q) = 1, \quad (7.4)$$

B being the associated 1-form and Q being the basic vector field corresponding to B. If X, Y, U, W are vector fields tangent to $\bar{M}$, then using (7.3) we can express (7.1) as follows:

$$\begin{aligned} g(\bar{R}(X,Y,U),W) = {} & p[g(Y,U)g(X,W) - g(X,U)g(Y,W)] \\ & + q[g(X,W)T(Y)T(U) + g(Y,U)T(X)T(W) \\ & - g(X,U)T(Y)T(W) - g(Y,W)T(X)T(U)] \\ & + g(B(X,W),B(Y,U)) - g(B(Y,W),B(X,U)). \end{aligned} \quad (7.5)$$

Since by hypothesis $\bar{M}$ is totally umbilical, (7.2) holds. Hence (7.5) takes the following form:

$$\begin{aligned} g(\bar{R}(X,Y,U),W) = (p+|\tau|^2)[g(Y,U)g(X,W) - g(X,U)g(Y,W)] \\ + q[g(X,W)T(Y)T(U) + g(Y,U)T(X)T(W) \\ - g(X,U)T(Y)T(W) - g(Y,W)T(X)T(U)]. \end{aligned} \tag{7.6}$$

or,

$$\begin{aligned} g(\bar{R}(X,Y,U),W) = l[g(Y,U)g(X,W) - g(X,U)g(Y,W)] \\ + m[g(X,W)T(Y)T(U) + g(Y,U)T(X)T(W) \\ - g(X,U)T(Y)T(W) - g(Y,W)T(X)T(U)], \end{aligned} \tag{7.7}$$

where

$$l = p + |\tau|^2 = \frac{r-2s}{(n-1)(n-2)} + |\tau|^2 \quad \text{and} \quad m = q = \frac{ns-r}{(n-1)(n-2)} \tag{7.8}$$

and $T(X) = B(X)$ by virtue of (7.4). Since in a conformally flat $A(PZS)_n$ $(n > 3)$, r cannot be zero, it follows from (7.7) that m cannot be zero. Again $T(X) = g(X,Q)$, where Q is a unit vector field. Comparing (7.7) with (2.8) we conclude that $\bar{M}$ is a manifold of quasi constant curvature with associated scalars l and m given by (7.8) and generator Q. This leads to the following theorem:

Theorem 7.1 *A totally umbilical hypersurface of a conformally flat $A(PZS)_n$ $(n > 3)$ is a manifold of quasi constant curvature.*

8 An example of an $A(PZS)_4$ which justify Theorem 5.1

Example 8.1 Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2, \tag{8.1}$$

where $(i, j = 1, 2, 3, 4)$. Here the only non-vanishing components of the Christoffel symbols and the curvature tensors are respectively:

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \qquad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}},$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}}, \qquad R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{\frac{2}{3}}}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are:

$$R_{11} = R_{22} = R_{33} = \frac{2}{3(x^4)^{\frac{2}{3}}}, \qquad R_{44} = -\frac{2}{3(x^4)^2},$$

Let us choose an arbitrary scalar function ϕ as $\phi = \frac{1}{(x^4)^2}$. Hence the non-vanishing components of the Z tensor and their covariant derivatives are respectively:

$$Z_{11} = Z_{22} = Z_{33} = \frac{5}{3(x^4)^{\frac{2}{3}}}, \qquad Z_{44} = \frac{1}{3(x^4)^2},$$

$$Z_{11,4} = Z_{22,4} = Z_{33,4} = -\frac{10}{9(x^4)^{\frac{5}{3}}}, \qquad Z_{44,4} = -\frac{2}{3(x^4)^3}.$$

It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is $\frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant. We shall now show that $(\mathbb{R}^4, g)$ is conformally flat. For this we shall prove that

$$C_{1221} = C_{1331} = C_{2332} = C_{1441} = C_{2442} = C_{3443} = 0,$$

as all other components of the conformal curvature tensor are zero automatically. Now,

$$\begin{aligned} C_{1221} &= R_{1221} - \frac{1}{2}[g_{11}R_{22} + g_{22}R_{11} - 2g_{12}R_{12}] + \frac{r}{(3)(2)}[g_{11}g_{22} - (g_{12})^2] \\ &= \frac{4}{9}(x^4)^{\frac{2}{3}} - \frac{1}{2}\left[(x^4)^{\frac{4}{3}}.\frac{2}{3}(x^4)^{-\frac{2}{3}} + (x^4)^{\frac{4}{3}}.\frac{2}{3}(x^4)^{-\frac{2}{3}}\right] + \frac{4}{(3)(2)3(x^4)^2}[(x^4)^{\frac{8}{3}}] \\ &= \frac{4}{9}(x^4)^{\frac{2}{3}} - \frac{1}{2}\left[\frac{4}{3}(x^4)^{\frac{2}{3}}\right] + \frac{2}{9}(x^4)^{\frac{2}{3}} \\ &= \frac{2}{3}(x^4)^{\frac{2}{3}} - \frac{2}{3}(x^4)^{\frac{2}{3}} = 0. \end{aligned}$$

By similar calculation it can be shown that

$$C_{1221} = C_{1331} = C_{2332} = C_{1441} = C_{2442} = C_{3443} = 0.$$

We shall now show that $\mathbb{R}^4$ is an $A(PZS)_n$. Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} -\frac{2}{3x^4} & \text{for i=4} \\ 0 & \text{otherwise,} \end{cases} \tag{8.2}$$

$$B_i(x) = \begin{cases} \frac{1}{x^4} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.8) reduces to the equations

$$Z_{11,4} = [A_4 + B_4]Z_{11} + A_1Z_{41} + A_1Z_{14}, \tag{8.4}$$

$$Z_{22,4} = [A_4 + B_4]Z_{22} + A_2Z_{42} + A_2Z_{24}, \tag{8.5}$$

$$Z_{33,4} = [A_4 + B_4]Z_{33} + A_3Z_{43} + A_3Z_{34}, \tag{8.6}$$

$$Z_{44,4} = [A_4 + B_4]Z_{44} + A_4Z_{44} + A_4Z_{44}, \tag{8.7}$$

since, for the other cases (1.8) holds trivially. By (8.2) and (8.3) we get the following relation for the right hand side (R.H.S.) and the left hand side (L.H.S.) of (8.4)

$$\begin{aligned}\text{R.H.S. of (8.4)} &= [A_4 + B_4]Z_{11} + A_1 Z_{41} + A_1 Z_{14} = [A_4 + B_4]Z_{11}\\ &= \Big(-\frac{2}{3x^4} + 0\Big)\frac{5}{3(x^4)^{\frac{2}{3}}} = -\frac{10}{9(x^4)^{\frac{5}{3}}}\\ &= Z_{11,4} = \text{L.H.S. of } \ (8.4).\end{aligned}$$

By similar argument it can be shown that (8.5), (8.6), (8.7) are true. So, $\mathbb{R}^4$ is an $A(PZS)_n$ whose scalar curvature is non-zero and non-constant.

We shall now show that this $(\mathbb{R}^4, g)$ is a quasi Einstein manifold. Let us choose the scalar functions a and b (the associated scalars) and the 1-form E as follows:

$$a = \frac{4}{3(x^4)^2}, \qquad b = -\frac{4}{(x^4)^2}, \qquad E_i(x) = \begin{cases} \frac{\left(x^4\right)^{\frac{2}{3}}}{\sqrt{6}} & \text{for i=1,2,3}\\ \frac{1}{\sqrt{2}} & \text{otherwise,}\end{cases}$$

at any point $x \in \mathbb{R}^4$. We can easily check that $(\mathbb{R}^4, g)$ is a quasi Einstein manifold which justify Theorem 5.1.

Acknowledgement. The authors are thankful to the referees for their valuable suggestions towards the improvement of this paper.

References

[1] Adati, T., Miyazawa, T.: *On a Riemannian space with recurrent conformal curvature.* Tensor, N. S. **18** (1967), 348–354.

[2] Binh, T. Q.: *On weakly symmetric Riemannian spaces.* Publ. Math. Debrecen **42** (1993), 103–107.

[3] Cartan, E.: *Sur une classes remarquable d'espaces de Riemannian.* Bull. Soc. Math. France **54** (1926), 214–264.

[4] Chaki, M. C., Gupta, B.: *On conformally symmetric spaces.* Indian J. Math. **5** (1963), 113–295.

[5] Chaki, M. C.: *On pseudo symmetric manifolds.* Ann. St. Univ. "Al I Cuza" Iasi **33** (1987), 53–58.

[6] Chaki, M. C.: *On pseudo Ricci symmetric manifolds.* Bulg. J. Phys. **15** (1988), 525–531.

[7] Chaki, M. C., Kawaguchi, T.: *On almost pseudo Ricci symmetric manifolds.* Tensor, N. S. **68** (2007), 10–14.

[8] Chen, B. Y., Yano, K.: *Hypersurfaces of a conformally flat spaces.* Tensor, N. S. **26** (1972), 318–322.

[9] Chern, S. S.: *On the curvature and characteristic classes of a Riemannian manifold.* Abh. Math. Sem. Univ. Hamburg **20** (1956), 117–126.

[10] De, U. C.: *On weakly symmetric structures on Riemannian manifolds.* Facta Univ. Ser. Mech. Automat. Control Robot. **3** (2003), 805–819.

[11] De, U. C., Bandyopadhyay, S.: *On weakly symmetric spaces.* Publ. Math. Debrecen **54** (1999), 377–381.

[12] De, U. C., Bandyopadhyay, S.: *On weakly symmetric spaces.* Acta Math. Hungarica **83** (2000), 205–212.

[13] De, U. C., Gazi, A. K.: *On almost pseudo symmetric manifolds.* Annales Univ. Sci. Budapest. **51** (2008), 53–68.

[14] De, U. C., Gazi, A. K.: *On almost pseudo conformally symmetric manifolds.* Demonstratio Mathematica **4** (2009), 869–886.

[15] De, U. C., Gazi, A. K.: *On conformally flat almost pseudo Ricci symmetric manifolds.* Kyungpook Math. J. **49** (2009), 507–520.

[16] De, U. C., Gazi, A. K.: *On pseudo Ricci symmetric manifold.* Analele Stiintifice Ale Universitatii "Al. I. Cuza" Din IASI (S. N.), Matematica **58** (2012), 209–222.

[17] De, U. C., Sengupta, J.: *On a weakly symmetric Riemannian manifold admitting a special type of semi-symmetric metric connection.* Novi Sad. J. Math. **29** (1999), 89–95.

[18] De, A., Őzgűr, C., De, U. C.: *On conformally flat almost pseudo-Ricci symmetric space times.* Int. J. Theor. Phys. **51** (2012), 2878–2887.

[19] Eisenhart, L. P.: Riemannian Geometry. *Princeton University Press*, Princeton, 1949.

[20] Gray, A.: *Einstein-like manifolds which are not Einstein.* Geom. Dedicata **7** (1978), 259–280.

[21] Hinterleitner, I., Mikeš, J.: *Geodesic mappings onto Weyl manifolds.* In: Proc. 8th Int. Conf. on Appl. Math. (APLIMAT 2009) **8** (2009), 423–430.

[22] Hui, S. K.: *On weakly W_3-symmetric manifolds.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **50** (2011), 53–71.

[23] Kobayashi, S., Nomizu, K.: Foundation of Differential Geometry, Vol. I. *Interscience Publishers*, New York, 1963.

[24] Mantica, C. A., Molinari, L. G.: *Weakly Z-Symmetric manifolds.* Acta Math. Hungarica **135** (2012), 80–96.

[25] Mantica, C. A., Suh, Y. J.: *Pseudo-Z-symmetric Riemannian manifolds with harmonic curvature tensors.* Int. J. Geom. Meth. Mod. Phys. **9** (2012), 1250004 1–21.

[26] Mantica, C. A., Suh, Y. J.: *Recurrent Z forms on Riemannian and Kaehler manifolds.* Int. J. Geom. Meth. Mod. Phys. **9** (2012), 1250059 1–26.

[27] Mikeš, J.: *On geodesic mappings of Ricci-2-symmetric Riemannian spaces.* Mat. Zametki **28** (1980), 313–317, Transl.: Math. Notes **28** (1980), 922–924.

[28] Mikeš, J.: *Geodesic mappings of affine-connected and Riemannian spaces. Geometry, 2.* J. Math. Sci. **78** (1996), 311–333.

[29] Mikeš, J., Hinterleitner, I., :, Vanžurová, A. Geodesic mappings and some generalizations. *Palacký University*, Olomouc, 2009.

[30] Mikeš, J., Chudá, H.: *On geodesic mappings with certain initial conditions.* Acta Math. Acad. Paedagog. Nyházi. (N.S.) **26** (2010), 337–341.

[31] O'Neill, B.: Semi-Riemannian Geometry with Applications to the Relativity. *Academic Press*, New York–London, 1983.

[32] Ozen, F., Altay, S.: *Weakly and pseudo-symmetric Riemannian spaces.* Indian J. Pure appl. Math. **33** (2002), 1477–1488.

[33] Ozen, F., Altay, S.: *On weakly and pseudo-concircular symmetric structures on a Riemannian manifold.* Acta Univ. Palack. Olomuc., Fac. rer. nat., Math. **47** (2008), 129–138.

[34] Prvanović, M.: *On weakly symmetric Riemannian manifolds.* Publ. Math. Debrecen **46** (1995), 19–25.

[35] Schouten, J. A.: Ricci-Calculus, An Introduction to Tensor Analysis and its Geometrical Applications. *Springer-Verlag*, Berlin–Göttingen–Heidelberg, 1954.

[36] Sinyukov, N. S.: Geodesic Mappings of Riemannian Spaces. *Nauka*, Moscow, 1979, (in Russian).

[37] Sen, R. N., Chaki, M. C.: *On curvature restrictions of a certain kind of conformally flat Riemannian space of class one*. Proc. Nat. Inst. Sci. India **33**, (part I) (1967), 100–102.

[38] Takeno, H., Ikeda, M.: *Theory of spherically symmetric space-times. VII. Space-times with corresponding geodesics*. J. Sci. Hiroshima Univ. **A 17**, 1 (1953), 75–81.

[39] Tamássy, L., Binh, T. Q.: *On weakly symmetric and weakly projectively symmetric Riemannian manifolds*. Colloq. Math. Soc. Janos Bolyai **56** (1989), 663–670.

[40] Tamássy, L., Binh, T. Q.: *On weak symmetries of Einstein and Sasakian manifolds.* Tensor, N. S. **53** (1993), 140–148.

[41] Walker, A. G.: *On Ruse's space of recurrent curvature*. Proc. London Math. Soc. **52** (1950), 36–54.

[42] Yano, K., Kon, M.: Structures on manifolds. *World Scientific*, Singapore, 1984, 418–421.

[43] Yilmaz, H. B.: *On decomposable almost pseudo conharmonically symmetric manifolds.* Acta Univ. Palacki. Olomuc. Fac. rer. nat., Math. **51** (2012), 111–124.