An Alternative Form
of the Functional Equation
for Riemann's Zeta Function, II

Andrea OSSICINI

Via delle Azzorre 352/D2, Roma, Italy
e-mail: andrea.ossicini@yahoo.it

Abstract

This paper treats about one of the most remarkable achievements by Riemann, that is the symmetric form of the functional equation for $\zeta(s)$. We present here, after showing the first proof of Riemann, a new, simple and direct proof of the symmetric form of the functional equation for both the Eulerian Zeta function and the alternating Zeta function, connected with odd numbers. A proof that Euler himself could have arranged with a little step at the end of his paper "Remarques sur un beau rapport entre les séries des puissances tant direct que réciproches". This more general functional equation gives origin to a special function, here named $\Im(s)$ which we prove that it can be continued analytically to an entire function over the whole complex plane using techniques similar to those of the second proof of Riemann. Moreover we are able to obtain a connection between Jacobi's imaginary transformation and an infinite series identity of Ramanujan. Finally, after studying the analytical properties of the function $\Im(s)$, we complete and extend the proof of a Fundamental Theorem, both on the zeros of Riemann Zeta function and on the zeros of Dirichlet Beta function, using also the Euler–Boole summation formula.

Key words: Riemann Zeta, Dirichlet Beta, generalized Riemann hypothesis, series representations

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1 Introduction

In [14] we introduced a special function, named $A(s)$, which is

$$A(s) = \frac{\Gamma(s)\zeta(s)L(s)}{\pi^s} \quad \text{with } s \in \boldsymbol{C}. \tag{1.1}$$

where $\Gamma(s)$ denotes Euler's Gamma function, $\zeta(s)$ denotes the Riemann Zeta function and $L(s)$ denotes Dirichlet's (or Catalan's) Beta function.

Let us remember that the Gamma function can be defined by the Euler's integral of the second kind [22, p. 241]:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt = \int_0^1 (\log 1/t)^{s-1}\, dt \qquad (\Re(s) > 0)$$

and also by the following Euler's definition [22, p. 237]:

$$\Gamma(s) = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{s(s+1)(s+2)\ldots(s+n-1)}\, n^s.$$

The Riemann Zeta function is defined by ([17, pp. 96–97], see Section 2.3):

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^\infty \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\[2ex] \displaystyle\left(1 - 2^{1-s}\right)^{-1} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0,\ s \neq 1) \end{cases}$$

which can be indeed analytically continued to the whole complex s-plane except for a simple pole at $s = 1$ with residue 1.

The Riemann Zeta function $\zeta(s)$ plays a central role in the applications of complex analysis to number theory.

The number-theoretic properties of $\zeta(s)$ are exhibited by the following result as *Euler's product formula*, which gives a relationship between the set of primes and the set of positive integers:

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1} \qquad (\Re(s) > 1),$$

where the product is taken over all primes.

It is an analytic version of the fundamental theorem of arithmetic, which states that every integer can be factored into primes in an essentially unique way.

Euler used this product to prove that the sum of the reciprocals of the primes diverges.

The Dirichlet Beta function, also known as Dirichlet's L function for the nontrivial character modulo 4, is defined, practically for $\Re(s) > 0$, by:

$$L(s) = L(s, \chi_4) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^s}$$

and it does not possess any singular point.

The $L(s)$ function is also connected to the theory of primes which may perhaps be best summarized by

$$L(s) = \prod_{p\equiv 1 \bmod 4} \left(1 - p^{-s}\right)^{-1} \prod_{p\equiv 3 \bmod 4} \left(1 - p^{-s}\right)^{-1} = \prod_{p \text{ odd}} \left(1 - (-1)^{\frac{p-1}{2}} p^{-s}\right)^{-1},$$

where the products are taken over primes and the rearrangement of factors is permitted because of an absolute convergence.

In [14] we have also proved the following identity:

$$A(s) = A(1-s) \tag{1.2}$$

and we have used the functional equation of $L(s)$ to rewrite the functional equation (1.2) in Riemann's well known functional equation for Zeta:

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) \tag{1.3}$$

or equivalently to

$$\zeta(1-s) = 2 \left(2\pi\right)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s).$$

This approach is the motivation for saying that the following symmetrical formulation:

$$\pi^{-s}\Gamma(s)\,\zeta(s)\,L(s) = \pi^{-(1-s)}\Gamma(1-s)\,\zeta(1-s)\,L(1-s).$$

is an alternative form of the functional equation for Riemann's Zeta Function.

2 The origin of the symmetric form of the functional equation for the Eulerian Zeta and for the alternating Zeta, connected with odd numbers

Riemann gives two proofs of the functional equation (1.3) in his paper [15], and subsequently he obtains the symmetric form by using two basic identities of the factorial function, that are Legendre's duplication formula [13], which was discovered in 1809 and was surely unknown to Euler:

$$\Gamma(s)\,\Gamma\left(s + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2s-1}}\,\Gamma(2s)$$

and Euler's complement formula:

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Riemann rewrites the functional equation (1.3) in the form [6, pp. 12–15]:

$$\zeta(s) = \frac{2^s \pi^{s-1}}{\sqrt{\pi}} 2^{-s} \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right)} \zeta(1-s)$$

and using the simplification $\pi^{s-1}\sqrt{\pi} = \frac{\pi^{-(1-s)/2}}{\pi^{-s/2}}$, he obtains the desired formula:

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2} \zeta(1-s).$$

Now this property induced Riemann to introduce, in place of $\Gamma(s)$, the integral $\Gamma\left(\frac{s}{2}\right)$ and at the end, for convenience, to define the ξ function as:

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2.1}$$

In this way $\xi(s)$ is an entire function and satisfies the simple functional equation:

$$\xi(s) = \xi(1-s). \tag{2.2}$$

This shows that $\xi(s)$ is symmetric around the vertical line $\Re(s) = \frac{1}{2}$.

In Remark 2 of [14] we stated that Euler himself could have proved the identity (1.2) using three reflection formulae of the $\zeta(s)$, $L(s)$ and $\Gamma(s)$, all well-known to him.

Here we present the simplest and direct proof based on the astonishing conjectures, that are Euler's main results in his work "Remarques sur un beau rapport entre les séries des puissances tant direct que réciproches" [8].

Euler writes the following functional equations:

$$\frac{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + 5^{n-1} - 6^{n-1} + \cdots}{1 - 2^{-n} + 3^{-n} - 4^{-n} + 5^{-n} - 6^{-n} + \cdots}$$
$$= -\frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)(2^n - 1)}{(2^{n-1} - 1)\pi^n}\cos\left(\frac{n\pi}{2}\right)$$

and

$$\frac{1 - 3^{n-1} + 5^{n-1} - 7^{n-1} + \cdots}{1 - 3^{-n} + 5^{-n} - 7^{-n} + \cdots} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)(2^n)}{\pi^n}\sin\left(\frac{n\pi}{2}\right)$$

and concludes his work by proving that those conjectures are valid for positive and negative integral values as well as for fractional values of n.

In modern notation we have, with $s \in \boldsymbol{C}$:

$$\frac{\eta(1-s)}{\eta(s)} = -\frac{(2^s - 1)}{\pi^s(2^{s-1} - 1)}\Gamma(s)\cos\left(\frac{s\pi}{2}\right) \tag{2.3}$$

and

$$\frac{L(1-s)}{L(s)} = \frac{2^s}{\pi^s}\Gamma(s)\sin\left(\frac{s\pi}{2}\right). \tag{2.4}$$

(2.3) represents the functional equation of Dirichlet's Eta function, which is defined for $\Re(s) \succ 0$ through the following alternating series:

$$\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}.$$

This function $\eta(s)$ is one simple step removed form $\zeta(s)$ as shown by the relation:

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s).$$

Thus (2.3) is easily manipulated into relation (1.3).

The (2.4) is the functional equation of Dirichlet's L function.

That being stated, multiplying (2.3) by (2.4) we obtain:

$$\frac{\eta(1-s)}{\eta(s)} \cdot \frac{L(1-s)}{L(s)} = \frac{(1-2^s)\, 2^{s-1}\, [\Gamma(s)]^2}{\pi^s\, (2^{s-1}-1)\, \pi^{s-1}} \cdot \frac{2\sin\left(\frac{s\pi}{2}\right)\cos\left(\frac{s\pi}{2}\right)}{\pi}.$$

Considering the duplication formula of $\sin(s\pi)$ and Euler's complement formula we have:

$$\frac{\eta(1-s)}{\eta(s)} \cdot \frac{L(1-s)}{L(s)} = \frac{(1-2^s)\, \pi^{1-s}}{(1-2^{1-s})\, \pi^s} \cdot \frac{\Gamma(s)}{\Gamma(1-s)}.$$

Shortly and ordering we obtain the following remarkable identity:

$$\frac{\left(1-2^{1-s}\right)}{\pi^{1-s}} \cdot \Gamma(1-s)\, \eta(1-s)\, L(1-s) = \frac{\left(1-2^s\right)}{\pi^s} \cdot \Gamma(s)\, \eta(s)\, L(s). \qquad (2.5)$$

This is unaltered by replacing $(1-s)$ by s.

3 The special function $\Im(s)$ and its integral representation

At this stage, let us introduce the following special function[1]:

$$\Im(s) = \frac{(1-2^s)\, \Gamma(s)\eta(s)L(s)}{\pi^s} = \frac{(1-2^s)\left(1-2^{1-s}\right)\Gamma(s)\zeta(s)L(s)}{\pi^s}. \qquad (3.1)$$

It is evident that from (1.1) one has:

$$\Im(s) = (1-2^s)\left(1-2^{1-s}\right)\, A(s).$$

This choice is based upon the fact that $\Im(s)$ is an entire function of s, hence it has no poles and satisfies the simple functional equation:

$$\Im(s) = \Im(1-s). \qquad (3.2)$$

The poles at $s = 0, 1$, respectively determined by the Gamma function $\Gamma(s)$ and by the Zeta function $\zeta(s)$ are cancelled by the term $(1-2^s) \cdot \left(1-2^{1-s}\right)$.

Now by using the identities [5, chap. X, p. 355, 10.15]:

$$\Gamma(s)\, a^{-s} = \int_0^\infty x^{s-1} e^{-ax}\, dx \equiv M_s\left\{e^{-ax}\right\} \qquad (3.3)$$

where M_s, denotes the Mellin transform and

$$\sum_m e^{-m^2 x} = \frac{1}{2}\left[\theta_3\left(0\,|ix/\pi\right) - 1\right] \qquad (3.4)$$

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[1] The letter $\Im$, called E reversed, is a letter of the Cyrillic alphabet and is the third last letter of the Russian alphabet.

where $\theta_3\left(z\left|\tau\right.\right)$ is one of the four theta functions, introduced by of Jacobi [22, chap. XXI] and the summation variable m is to run over all positive integers, we derive the following integral representation of $\Im(s)$ function:

$$\Im(s) = \frac{(1-2^s)}{2}\frac{\left(1-2^{1-s}\right)}{2}\int_0^\infty \left[\theta_3^2\left(0\left|ix/\pi\right.\right)-1\right]\left(\frac{x}{\pi}\right)^s\frac{dx}{x} \qquad (3.5)$$

Indeed combining the following two Mellin transforms:

$$\Gamma(s)\zeta(2s) = M_s\left\{\frac{1}{2}\left[\theta_3\left(0\left|ix/\pi\right.\right)-1\right]\right\} \qquad \left(\Re(s) > \frac{1}{2}\right)$$

and

$$\Gamma(s)\left[L(s)\zeta(s)-\zeta(2s)\right] = M_s\left\{\frac{1}{4}\left[\theta_3\left(0\left|ix/\pi\right.\right)-1\right]^2\right\} \qquad (\Re(s) > 1),$$

the former is immediately obtained from Eqs. (3.3) and (3.4) and the latter is obtained integrating term by term the following remarkable identity, obtained from an identity by Jacobi [11] and the result[2] $\theta_3^2\left(0\left|\tau\right.\right) = 2K/\pi$ [22, p. 479]:

$$\frac{1}{4}\left[\theta_3^2\left(0\left|ix/\pi\right.\right)-1\right] = \sum_\ell(-1)^{(\ell-1)/2}\left[e^{\ell x}-1\right]^{-1}$$

(here the sum is to expanded as a geometric series in $e^{-\ell x}$:

$$e^{-\ell x}+e^{-2\ell x}+e^{-3\ell x}+e^{-4\ell x}+e^{-5\ell x}+\ldots = \left[e^{\ell x}-1\right]^{-1}$$

and the summation variable ℓ runs over all positive odd integers), thus we are in the position to determine the integral representation (3.5) for the $\Im(s)$ function, by the linearity property of Mellin transformation, from the following identity:

$$\begin{aligned}\Im(s) &= \frac{(1-2^s)\left(1-2^{1-s}\right)\Gamma(s)\,\zeta(s)\,L(s)}{\pi^s}\\ &= \frac{(1-2^s)\left(1-2^{1-s}\right)}{\pi^s}\left[M_s\left\{\frac{1}{4}\left[\theta_3(0\left|ix/\pi\right.)-1\right]^2\right\}\right.\\ &\quad \left.- M_s\left\{\frac{1}{2}\left[\theta_3(0\left|ix/\pi\right.)-1\right]\right\}\right]\\ &= \frac{(1-2^s)\left(1-2^{1-s}\right)}{\pi^s}M_s\left\{\frac{1}{4}\left[\theta_3^2(0\left|ix/\pi\right.)-1\right]\right\} \quad (\Re(s) > 1).\quad (3.6)\end{aligned}$$

Now we start from (3.5) to give an independent proof of (3.2) that does not use (2.3) and (2.4), adopting techniques similar to Riemann's ones we use the following fundamental transformation formula for $\theta_3\left(z\left|\tau\right.\right)$:

$$\theta_3(z\left|\tau\right.) = (-i\tau)^{-1/2}\exp\left(z^2/\pi\,i\tau\right)\cdot\theta_3\left(\frac{z}{\tau}\left|-\frac{1}{\tau}\right.\right) \qquad (3.7)$$

---

[2] K denotes the complete elliptic integral of the first kind of modulus k.

where $(-i\tau)^{-1/2}$ is to be interpreted by the convention $|\arg(-i\tau)| < \frac{1}{2}\pi$ [22, p. 475].

In particular we obtain that:

$$\theta_3^2\left(0\left|\frac{i\,x}{\pi}\right.\right) = \frac{\pi}{x}\theta_3^2\left(0\left|\frac{i\,\pi}{x}\right.\right). \tag{3.8}$$

We then rewrite the integral that appears in (3.5) as:

$$\int_0^\pi \left[\theta_3^2(0\,|ix/\pi) - 1\right]\cdot\left(\frac{x}{\pi}\right)^s\frac{dx}{x} + \int_\pi^\infty \left[\theta_3^2(0\,|ix/\pi) - 1\right]\cdot\left(\frac{x}{\pi}\right)^s\frac{dx}{x}$$

$$= \int_0^\pi \left[\theta_3^2(0\,|ix/\pi)\right]\cdot\left(\frac{x}{\pi}\right)^s\frac{dx}{x} - \frac{1}{s} + \int_\pi^\infty \left\{\left[\theta_3^2(0\,|ix/\pi) - 1\right]\right\}\cdot\left(\frac{x}{\pi}\right)^s\frac{dx}{x}$$

and use the change of variable $\frac{ix}{\pi} \to \frac{i\pi}{x}$ and the (3.8) to find:

$$\int_0^\pi \left[\theta_3^2(0\,|ix/\pi)\right]\cdot\left(\frac{x}{\pi}\right)^s\frac{dx}{x} = \int_\pi^\infty \left[\theta_3^2(0\,|i\pi/x)\right]\cdot\left(\frac{\pi}{x}\right)^s\frac{dx}{x}$$

$$= \int_\pi^\infty \left[\theta_3^2(0\,|ix/\pi)\right]\cdot\left(\frac{x}{\pi}\right)^{1-s}\frac{dx}{x} = -\frac{1}{1-s} + \int_\pi^\infty \left[\theta_3^2(0\,|ix/\pi) - 1\right]\cdot\left(\frac{x}{\pi}\right)^{1-s}\frac{dx}{x}.$$

Therefore:

$$\Im(s) = \frac{(1-2^s)}{2}\frac{\left(1-2^{1-s}\right)}{2}.$$

$$\cdot\left\{\frac{1}{s\,(s-1)} + \int_\pi^\infty \left[\theta_3^2(0\,|ix/\pi) - 1\right]\cdot\left[\left(\frac{x}{\pi}\right)^s + \left(\frac{x}{\pi}\right)^{1-s}\right]d\log x\right\} \tag{3.9}$$

which is manifestly symmetrical under $s \to 1-s$, and analytic since $\theta_3\left(0\left|\frac{ix}{\pi}\right.\right)$ decreases exponentially as $x \to \infty$.

This concludes the proof of the functional equation and the analytic continuation of $\Im(s)$, assuming the identity (3.7), due to Jacobi.

4 Jacobi's imaginary transformation and an infinite series identity of Ramanujan

The fundamental transformation formula of Jacobi for $\theta_3(z\,|\tau)$:

$$\theta_3(z\,|\tau) = (-i\tau)^{-1/2}\exp\left(z^2/\pi\,i\tau\right)\cdot\theta_3\left(\frac{z}{\tau}\left|-\frac{1}{\tau}\right.\right)$$

where the squares root is to be interpreted as the principal value; that is, if $w = re^{i\theta}$ where $0 \leq \theta \leq 2\pi$, then $w^{1/2} = r^{1/2}e^{i\theta/2}$ and the infinite series identity of Ramanujan [3, Entry 11, p. 258]:

$$\alpha\left\{\frac{1}{4}\sec(\alpha n) + \sum_{k=1}^\infty \chi(k)\frac{\cos(\alpha\,nk)}{e^{\alpha^2 k} - 1}\right\} = \beta\left\{\frac{1}{4} + \frac{1}{2}\sum_{k=1}^\infty \frac{\cosh(2\beta nk)}{\cosh(\beta^2 k)}\right\}$$

with $\alpha, \beta \succ 0$, $\alpha\beta = \pi$, $n \in \Re$, $|n| \prec \beta/2$ and with

$$\chi(k) = \begin{cases} 0 & \text{for } k \text{ even} \\ 1 & \text{for } k \equiv 1 \bmod 4 \\ -1 & \text{for } k \equiv 3 \bmod 4 \end{cases}$$

can be derived from the following Poisson summation formula (see [2, pp. 7–11] and [14, Appendix]):

$$\frac{1}{2}f(0) + \sum_{k=1}^{\infty} f(k) = \int_0^{\infty} f(x)\,dx + 2\sum_{k=1}^{\infty}\int_0^{\infty} f(x)\cos(2k\pi x)\,dx.$$

From Jacobi's Lambert series formula for $\theta_3^2(0\,|\tau)$:

$$\theta_3^2(0\,|\tau) - 1 = 4\sum_{\ell}(-1)^{(\ell-1)/2}q^{\ell}\left(1 - q^{\ell}\right)^{-1},$$

where ℓ is to run over all positive odd integers, we have again with $q = \exp(i\pi\tau)$, and $\tau = ix/\pi$:

$$\frac{1}{4}\left[\theta_3^2(0\,|ix/\pi) - 1\right] = \sum_{\ell}(-1)^{(\ell-1)/2}\left[e^{\ell x} - 1\right]^{-1}.$$

Now

$$\sum_{\ell}(-1)^{(\ell-1)/2}\left[e^{\ell x} - 1\right]^{-1} = \sum_{m=1}^{\infty}\chi(m)\frac{1}{e^{mx} - 1}$$

where still

$$\chi(m) = \begin{cases} 0 & \text{for } m \text{ even} \\ 1 & \text{for } m \equiv 1 \bmod 4 \\ -1 & \text{for } m \equiv 3 \bmod 4 \end{cases}$$

and therefore

$$\frac{1}{4}\left[\theta_3^2(0\,|ix/\pi) - 1\right] = \sum_{m=1}^{\infty}\chi(m)\frac{1}{e^{mx} - 1} \tag{4.1}$$

For $n = 0$ the infinite series identity of Ramanujan reads

$$\alpha\left\{\frac{1}{4} + \sum_{k=1}^{\infty}\chi(k)\frac{1}{e^{\alpha^2 k} - 1}\right\} = \beta\left\{\frac{1}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\cosh(\beta^2 k)}\right\}.$$

Replacing $\cosh(x)$ by the exponential functions, expanding the geometric series and rearranging the sums we have

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\cosh(\beta^2 k)} = \sum_{m=1}^{\infty}\chi(m)\frac{1}{e^{\beta^2 m} - 1}.$$

Now we substitute $\alpha = \sqrt{x}$, $\beta = \pi/\sqrt{x}$ and we obtain:

$$\left\{ \frac{1}{4} + \sum_{k=1}^{\infty} \chi(k) \frac{1}{e^{xk} - 1} \right\} = \frac{\pi}{x} \left\{ \frac{1}{4} + \sum_{m=1}^{\infty} \chi(m) \frac{1}{e^{(\pi^2/x)m} - 1} \right\}.$$

Finally, with the relation (4.1) we establish the following transformation of $\theta_3^2 \left(0 \left| \frac{ix}{\pi} \right. \right)$:

$$\theta_3^2 \left(0 \left| \frac{i\,x}{\pi} \right. \right) = \frac{\pi}{x} \theta_3^2 \left(0 \left| \frac{i\pi}{x} \right. \right).$$

This last transformation is also an immediate consequence of the fundamental transformation formula of Jacobi for $\theta_3(z \,|\, \tau)$.

In this way we have obtained an amazing connection between the Jacobi imaginary transformation and the infinite series identity of Ramanujan.

5   The properties of the function $\Im(s)$

In this section we remark the following fundamental properties of the special function $\Im(s)$ with $s = \sigma + it$:

(a) $\Im(s) = \Im(1 - s)$

(b) $\Im(s)$ is an entire function and $\Im(s) = \overline{\Im(\overline{s})}$

(c) $\Im \left(\frac{1}{2} + it \right) \in \Re$

(d) $\Im(0) = \Im(1) = -\frac{\log 2}{4}$

(e) if $\Im(s) = 0$, then $0 \leq \sigma \leq 1$

(f) $\Im(s) < 0$ for all $s \in \Re$.

Outline of proof:

Using the topics developed at the end of Sections 2 and 3, the functional equation (a) follows.

Regarding (b), the second expression in the definition (3.1) shows at once that $\Im(s)$ is holomorphic for $\sigma \geq 0$, since the simple pole of $\Gamma(s)$ at $s = 0$ and the simple pole of $\zeta(s)$ at $s = 1$ are removed by the factors $(1 - 2^s)$ and $\left(1 - 2^{1-s}\right)$, and there are no poles for $\sigma \geq 0$, but the (a) implies $\Im(s)$ holomorphic on all $\boldsymbol{C}$.

The second part of (b) follows from the fact that $\Im(s)$ is real on the real line, thus $\Im(s) - \overline{\Im(\overline{s})}$, is an analytic function vanishing on the real line, hence zero since the zeros of an analytic function which is not identically zero can have no accumulation point.

We note that $s = \frac{1}{2} + it$ where t is real, then $\overline{s}$ and $1 - s$ coincide, so this implies (c).

The known values $L(1) = \frac{\pi}{4}$, $\eta(1) = \log 2$ and $\lim_{s \to 1} \frac{(1-2^s)}{\pi^s} \Gamma(s) = -\frac{1}{\pi}$ imply (d) for $\Im(1)$ and the functional equation (a) then gives the result for $\Im(0)$.

Since the Gamma function has no zeros and since the Dirichlet Beta function and the Riemann Zeta function have respectively an Euler product (see §1.

Introduction or [9, p. 53 and p. 40]):

$$L(s) = \prod_{\substack{p \\ \text{primeodd}}} \left(1 - (-1)^{\frac{p-1}{2}} p^{-s}\right)^{-1}; \qquad \zeta(s) = \prod_{\text{prime}}^{p} \left(1 - p^{-s}\right)^{-1}$$

which shows that they are non-vanishing in the right half plane $\Re(s) > 1$, the function $\Im(s)$ has no zeros in $\Re(s) > 1$, and by functional equation (a), it also has non zeros in $\Re(s) < 0$.

Thus all the zeros have their real parts between 0 and 1 (including the extremes) and this proves (e).

Finally, to prove (f) first we note from the following integral representations [7, p. 1, p. 32 and p. 35]:

$$\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x}\, dx; \quad L(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + e^{-x}}\, dx;$$

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1}\, dx \quad (\Re(s) > 0)$$

that $\Gamma(s)$, $L(s)$, $\eta(s)$ are positives for all $s \in \Re$, $s > 0$.

Then combining this with the negative factor $\frac{1-2^s}{\pi^s}$ for $s > 0$ the definition (3.1) proves (f) for $s > 0$, $s \neq 0$ and combining this with (d) then it proves (f) for $s \geq 0$, whence the functional equation (a) shows that (f) holds for all $s \in \Re$.

6 The zeros of the entire function $\Im(s)$ and an estimate for the number of these in the critical strip $0 \leq \sigma \leq 1$

We summarized and extended the results of the previous section in the following theorem:

Fundamental Theorem (i) *The zeros of $\Im(s)$ (if any exits) are all situated in the strip $0 \leq \sigma \leq 1$ and lie symmetrically about the lines $t = 0$ and $\sigma = \frac{1}{2}$.*

(ii) *The zeros of $\Im(s)$ are identical to the imaginary zeros of the factor $(1 - 2^s) \cdot (1 - 2^{1-s})$ and to the non-trivial zeros of the functions $L(s)$ and $\zeta(s)$; $\Im(s)$ has no zeros on the real axis.*

(iii) *The number $N(T)$ of zeros of $\Im(s)$ in the rectangle with $0 \leq \sigma \leq 1$, $0 \leq t \leq T$, when $T \to \infty$ satisfies:*

$$N(T) = \frac{T}{\pi} \log \frac{2T}{\pi e} + O(\log T)$$

where the notation $f(T) = O(g(T))$ means $\frac{f(T)}{g(T)}$ is bounded by a constant independent of T.

Proof To prove (i) the properties (a) and (e) are sufficient.

These properties together with (b) and (c) show that we may detect zeros of $\Im(s)$ on the line $\sigma = \frac{1}{2}$ by detecting sign changes, for example, in $\Im\left(\frac{1}{2} + it\right)$, so

it is not necessary to compute exactly the location of a zero in order to confirm that it is on this line.

Thus we compute

$$\Im\left(\frac{1}{2} + 5i\right) = -2.519281933\ldots \cdot 10^{-3}; \quad \Im\left(\frac{1}{2} + 7i\right) = +8.959203701\ldots \cdot 10^{-5}$$

we know that there is a zero of $\Im\left(\frac{1}{2} + it\right)$ with t between 5 and 7.

Indeed for $t = 6.0209489\ldots$, we have $\Im\left(\frac{1}{2} + 6.0209489\ldots i\right) = 0$ and this is the smallest zero of $\Im(s)$: a much smaller value than the one corresponding to $\zeta(s)$, that is $\zeta\left(\frac{1}{2} + 14.13472514\ldots i\right) = 0$.

To prove (ii) we have:

$$\Im(s) = h(s)\frac{\Gamma(s)\zeta(s)L(s)}{\pi^s}$$

where the imaginary zeros of the factor $h(s) = (1 - 2^s) \cdot \left(1 - 2^{1-s}\right)$ lie on the vertical lines $\Re(s) = 0$ and $\Re(s) = 1$.

We recall the following identity of the general exponential function $w = a^z$ ($a \neq 0$ is any complex number): $a^z = e^{z \log a}$; now, the function e^z assumes all values except zero, i.e. the equation $e^z = A$ is solvable for any nonzero complex number A.

If $\alpha = \arg A$, all solutions of the equation $e^z = A$ are given by the formula:

$$z = \log|A| + i\left(\alpha + 2k\pi\right), \quad k = 0, \pm 1, \pm 2, \ldots$$

In particular, if $e^z = 1$, we have $z = 2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$

Consequently, the imaginary roots of $h(s)$ are $s = \pm\frac{2\pi i k}{\log 2}$ and $s = 1 \pm \frac{2\pi i k}{\log 2}$ with $k \in N$, $k > 0$.

In addition from each of functional equations (2.3) and (2.4), exploiting the zeros of the trigonometric function *cosine* and *sine*, it is immediate to verify that:

$$\zeta(s) = 0 \quad \text{for } s = -2, -4, -6, -8, \ldots$$

and

$$L(s) = 0 \quad \text{for } s = -1, -3, -5, -7, \ldots.$$

These are the trivial zeros of the two Euler's Zeta functions $\zeta(s)$ and $L(s)$, that are cancelled by the singularities of the $\Gamma(s)$ function in the negative horizontal axis x.

We remember that the two last singularities at $s = 0$, 1, respectively determined by the $\Gamma(s)$ function and by $\zeta(s)$ function, are cancelled by real roots of factor $h(s)$.

We've still got the non-trivial zeros of the functions $\zeta(s)$ and $L(s)$, see Section 5 and at the end let's see also the property (f).

For the proof of (iii) we consider the fact that $\Im(s)$ is an entire function of s, hence it has no poles and the result (ii).

These properties can be then used to estimate $N(T)$ by calling upon the Argument Principle [10, pp. 68–70].

The Argument Principle is the following theorem of Cauchy:

Theorem 6.1 *Suppose the function $F(s)$ is analytic, apart from a finite number of poles, in the closure of a domain D bounded by a simple closed positively oriented Jordan curve C. Suppose further that $F(s)$ has no zeros or poles on C. Then the total number of zeros of $F(s)$ in D, minus the total number of poles of $F(s)$ in D, counted with multiplicities, is given by*

$$\frac{1}{2\pi i} \int_C \frac{F'(s)}{F(s)} \, ds = \frac{1}{2\pi} \Delta_C \arg F(s).$$

Here $\Delta_C \arg F(s)$ denotes the change of argument of $F(s)$ along C.

In addition we consider the following results obtained from the Stirling's formula [16] and Jensen's formula [10, pp. 49–50]:

Proposition 6.1 (Stirling's formula) *We have*

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + O\left(|s|^{-1}\right) \approx \left(s - \frac{1}{2}\right) \log s - s + O(1)$$

valid as $|s| \to \infty$, in the angle $-\pi + \delta < \arg s < \pi - \delta$, for any fixed $\delta > 0$.

Proposition 6.2 *Let f be a function which is analytic in a neighborhood of the disk $|z - a| < R$. Suppose $0 < r < R$ and that f has n zeros in the disk $|z - a| < r$. Let $M = \max \left| f\left(a + Re^{i\theta}\right)\right|$ and suppose that $|f(0)| \neq 0$. Then*

$$\left(\frac{R}{r}\right)^n \leq \frac{M}{|f(0)|}.$$

We begin considering the Theorem 6.1 for the function $\Im(s)$ in the region R, whose R is a rectangle in the complex plane with vertices at 2, $2 + iT$, $-1 + iT$ and -1 (see Fig. 1 and Appendix).

Let D be the rectangular path passing through these vertices in the anti-clockwise direction.

It was noted earlier that $\Im(s)$ is analytic everywhere, and has as its only zeros the imaginary zeros in the critical strip.

Hence the number of zeros in the region R, which is given by the equation

$$N(T) = \frac{1}{2\pi i} \int_D \Im'(s) / \Im(s) \, ds = \frac{1}{2\pi} \Delta_D \arg \Im(s)$$

and so

$$2\pi N(T) = \Delta_D \arg \Im(s).$$

Our study of $N(T)$ will therefore focus on the change of the argument of $\Im(s)$ as we move around the rectangle D. As we move along the base of this rectangle, there is no change in $\arg \Im(s)$, since $\Im(s)$ is real along this path and is never equal to zero.

We wish to show that the change in $\arg \Im(s)$ as s goes from $\frac{1}{2} + iT$ to $-1 + iT$ and then to -1 is equal to the change as s moves from 2 to $2 + iT$ to $\frac{1}{2} + iT$.
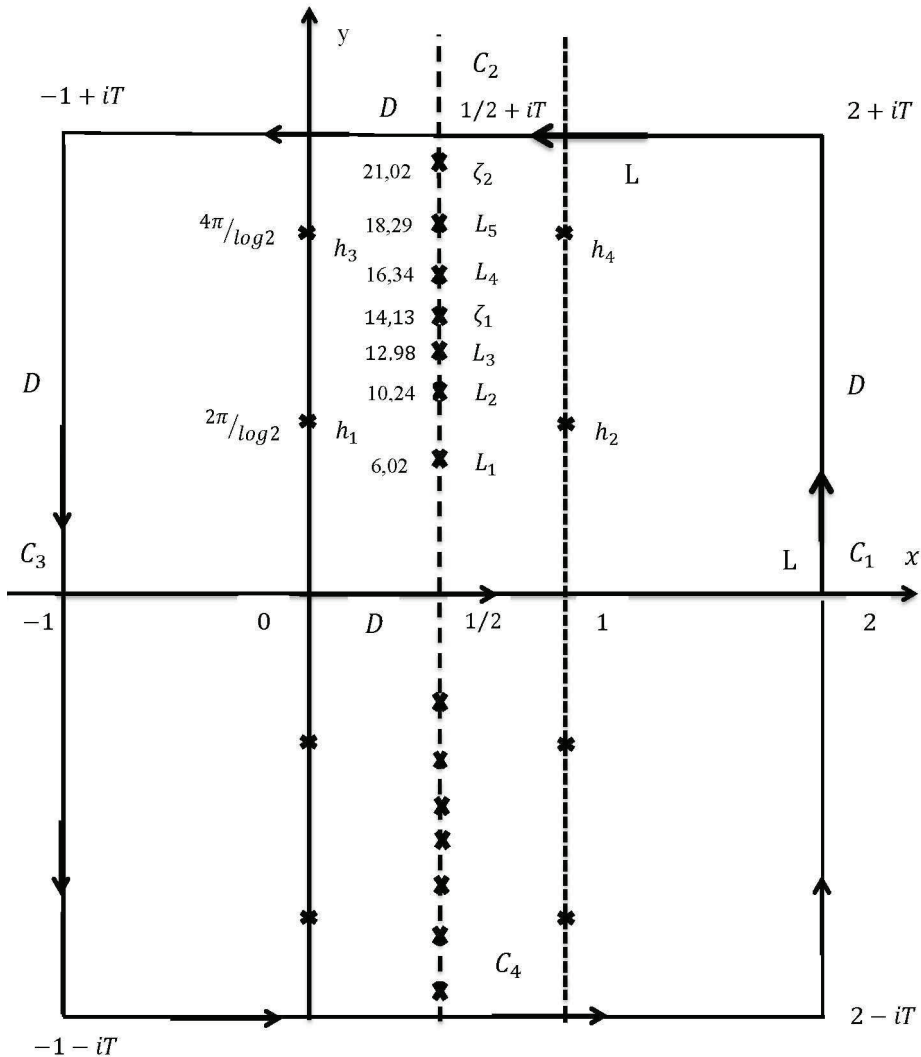
Figure 1 The complex zeros of the special function $\Im(s)$

Legend: zeros = ✗, $i = 1, 2, 3, \ldots$, ζ_i = zeros of $\zeta(s)$, L_i = zeros of $L(s)$, h_i = zeros of $h(s)$.

To see this we observe that

$$\Im(\sigma + it) = \Im(1 - \sigma - it) = \overline{\Im(1 - \sigma + it)}.$$

Hence the change in argument over the two paths will be the same.

If we define L to be the path from 2 to $2 + iT$ then $\frac{1}{2} + iT$, we have that

$$2\pi N(T) = 2\Delta_L \arg \Im(s) \quad \text{or} \quad \pi N(T) = \Delta_L \arg \Im(s).$$

We now recall the definition of $\Im(s)$ given by

$$\Im(s) = \frac{\left(1 - 2^s\right)\left(1 - 2^{1-s}\right)\Gamma(s)\zeta(s)L(s)}{\pi^s}$$

and consider the argument of each section of the right-hand-size in turn.

We have:

$$\Delta_L \arg\left[\left(1 - 2^s\right)\left(1 - 2^{1-s}\right)\right]$$

$$= \Delta_L \arg\left(1 - 2^s\right) + \Delta_L \arg\left(1 - 2^{1-s}\right) = 2\Delta_L \arg\left(1 - 2^{1-s}\right) = T\log 2 + O(1)$$

and

$$\Delta_L \arg \pi^{-s} = \Delta_L \arg \exp(-s\log\pi) = \Delta_L(-t\log\pi) = -T\log\pi.$$

The proof of this first result is provided in Appendix.

To consider $\Gamma(s)$ we call on Stirling's formula and also $\arg z = \Im\log z$, thus we have:

$$\Delta_L \arg\Gamma(s) = \Im\log\Gamma\left(\frac{1}{2} + iT\right) = \Im\left[iT\log\left(\frac{1}{2} + iT\right) - \frac{1}{2} - iT + O(1)\right]$$

or since

$$\log\left(\frac{1}{2} + iT\right) = \log\left|\frac{1}{2} + iT\right| + i\frac{\pi}{2} = \log\sqrt{\frac{1}{4} + T^2} + i\frac{\pi}{2} \approx \log T + O\left(\frac{1}{T}\right) + i\frac{\pi}{2}$$

$$\Delta_L \arg\Gamma(s) = T\log T - T + O(1).$$

The above arguments can then be combined giving

$$\pi N(T) = \Delta_L \arg\Im(s)$$

$$= \Delta_L \arg\left[\left(1 - 2^s\right)\left(1 - 2^{1-s}\right)\right]$$

$$+ \Delta_L \arg \pi^{-s} + \Delta_L \arg\Gamma(s) + \Delta_L \arg\zeta(s) + \Delta_L \arg L(s)$$

$$= T\left(\log 2 - \log\pi + \log T - 1\right) + O(1) + \Delta_L \arg\zeta(s) + \Delta_L \arg L(s)$$

$$= T\log\frac{2T}{\pi e} + \Delta_L \arg\zeta(s) + \Delta_L \arg L(s) + O(1).$$

Hence

$$N(T) = \frac{T}{\pi}\log\frac{2T}{\pi e} + R(T) + S(T) + O(1)$$

where

$$\pi\left[R(T) + S(T)\right] = \Delta_L \arg\zeta(s) + \Delta_L \arg L(s).$$

From this point, in order to prove the approximation for $N(T)$ initially claimed in (iii) it will be sufficient to show

$$R(T) = S(T) = O(\log T) \quad \text{as } T \to \infty. \tag{6.1}$$

First we need to know a bound for $\zeta(s)$ and $L(s)$ on vertical strips.

Let $s = \sigma + it$ where σ and t are real.

Proposition 6.3 *Let $0 < \delta < 1$. In the region $\sigma \geq \delta$, $t > 1$ we have*

$$\text{(A)} \ \ \zeta(\sigma + it) = O\left(t^{1-\delta}\right); \quad \text{(B)} \ \ L(\sigma + it) = O\left(t^{1-\delta}\right).$$

Proof Firstly we will deduce before the estimate (B). To achieve this goal we use the following formula of Euler–Boole summation[3], because it is used to explain the properties of alternating series and it is better suited than Euler–Maclaurin summation [4].

Let $0 \leq h \leq 1$ and a, m and n integers such $n > a$, $m > 0$ and $f^{(m)}(x)$ is absolutely integrable over $[a, n]$.

Then we have:

$$\sum_{j=a}^{n-1}(-1)^j f(j+h) = \frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(h)}{k!}\left((-1)^{n-1}f^{(k)}(n) + (-1)^a f^{(k)}(a)\right)$$

$$+ \frac{1}{2(m-1)!}\int_a^n f^{(m)}(x)\widetilde{E}_{m-1}(h-x)\,dx.$$

$E_n(x)$ are Euler polynomials given by the generating function:

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

and the periodic Euler polynomials $\tilde{E}_n(x)$ are defined by setting $\tilde{E}_n(x) = E_n(x)$ for $0 \leq x \prec 1$ and $\tilde{E}_n(x+1) = -\tilde{E}_n(x)$ for all other x.

Let N be a larger integer to be determined later.

If f is any smooth function, for $M > N$, in the formula of Euler–Boole summation above, with $a = N$, $m = 1$ and by taking the limit as $h \to 0$ we obtain:

$$\sum_{n=N}^{M-1}(-1)^n f(n) = \frac{1}{2}\left[(-1)^N f(N) + (-1)^{M-1}f(M)\right] + \frac{1}{2}\int_N^M \tilde{E}_0(-x)f'(x)\,dx$$

where $\tilde{E}_0(x) = sgn\left[\sin(\pi x)\right]$, that is a pieciewise constant periodic function.

Take $f(x) = (2x+1)^{-s}$, where initially $\Re(s) > 1$, and let $M \to \infty$.

We obtain:

$$L(s) - \sum_{n \prec N}\frac{(-1)^n}{(2n+1)^s} = \sum_{n=N}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

$$= \frac{1}{2}\left[(-1)^N(2N+1)^{-s}\right] - s\int_N^{\infty}\tilde{E}_0(-x)(2x+1)^{-s-1}\,dx.$$

The integral

$$s\int_N^{\infty}\tilde{E}_0(-x)(2x+1)^{-s-1}\,dx$$

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[3] NIST, Digital Library of Mathematical Functions, (forthcoming) http://dlmf.nist.gov/24.17

is absolutely convergent if $\sigma = \Re(s) > 0$, and since $\left|\tilde{E}_0(-x)\right| = 1$, we note that

$$\left| s \int_N^\infty (2x+1)^{-s-1}\, dx \right| < |s| \int_N^\infty (2x+1)^{-\sigma-1}\, dx$$

$$= \frac{|s|}{\sigma}(2N+1)^{-\sigma} \leq \left(1 + \frac{t}{\sigma}\right)(2N+1)^{-\sigma}$$

where we have used the triangle inequality $|s| \leq \sigma + t$.

Also

$$\left| \sum_{n \prec N} \frac{(-1)^n}{(2n+1)^s} \right| \leq \sum_{n \prec N} \frac{(-1)^n}{(2n+1)^\sigma} < \int_0^N (2x+1)^{-\sigma}\, dx = \frac{(2N+1)^{1-\sigma}}{1-\sigma} - \frac{1}{1-\sigma}.$$

Thus

$$|L(s)| = \left| \sum_{n \prec N} \frac{(-1)^n}{(2n+1)^s} + \frac{1}{2}\left[(-1)^N(2N+1)^{-s}\right] - s \int_N^\infty (2x+1)^{-s-1}\, dx \right|$$

$$\leq \frac{(2N+1)^{1-\sigma}}{1-\sigma} - \frac{1}{1-\sigma} + \frac{1}{2}\left[(-1)^N(2N+1)^{-\sigma}\right] + \left(1 + \frac{t}{\sigma}\right)(2N+1)^{-\sigma}$$

$$< \frac{(2N+1)^{1-\sigma}}{1-\sigma} + \left(\frac{3}{2} + \frac{t}{\sigma}\right)(2N+1)^{-\sigma}. \tag{6.2}$$

Assuming that $t > 1$, we may estimate this by taking N to be greatest integer less than $\left(\frac{t-1}{2}\right)$.

To see that this is the optimal choice of t, consider the two potentially largest terms in (6.2):

$$\frac{(2N+1)^{1-\sigma}}{1-\sigma} \quad \text{and} \quad \left(\frac{t}{\sigma}\right)(2N+1)^{-\sigma}.$$

If we take N to be approximately $\frac{t^\alpha - 1}{2}$ for some α, these are $\frac{t^{\alpha(1-\sigma)}}{(1-\sigma)}$ and $(\sigma)^{-1} \cdot t^{1-\alpha\sigma}$.

As α varies, one increases, the other decreases. Thus, we want to equate the exponents, so $\alpha(1-\sigma) = 1 - \alpha\sigma$, or $\alpha = 1$.

Taking $N \approx \frac{t-1}{2}$, we see that $L(s)$ is of the order $O\left(t^{1-\sigma}\right)$.

If $\sigma \geq \delta$ and $t > 1$, we see that $L(\sigma + it) = O\left(t^{1-\delta}\right)$ and thus (B) is proved.

To achieve the estimate (A) it is sufficient to use the same procedure, but in this case we recall the formula of Euler–Maclaurin, that is

$$\sum_{n=N}^M f(n) = \int_N^M f(x)\, dx + \frac{1}{2}f(N) + \frac{1}{2}f(M) + \int_N^M B_1(x - [x])f'(x)\, dx$$

where $B_1(x) = x - \frac{1}{2}$ is the first Bernouilli polynomial, $[x]$ is the greatest integer and take $f(x) = x^{-s}$, where initially $\Re(s) > 1$, and let $M \to \infty$.

In this case, at the end, we obtain

$$|\zeta(s)| \leq \frac{(N)^{1-\sigma}}{1-\sigma} + \frac{(N)^{1-\sigma}}{t} + \left(\frac{1}{2} + \frac{t}{2\sigma}\right)(N)^{-\sigma}.$$

Taking $N \approx t$, we see that, if $\sigma \geq \delta$ and $t > 1$, $\zeta(\sigma + it) = O\left(t^{1-\delta}\right)$. Consequently (A) is proved. $\qquad\square$

Finally we will prove (6.1), that is the integrals

$$\Im\left(\int_2^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)} \, ds + \int_2^{\frac{1}{2}+iT} \frac{L'(s)}{L(s)} \, ds\right) = O(\log T).$$

Firstly we note that $\zeta(s)$ and $L(s)$ are holomorphic and non-vanishing in the half plane $\Re(s) \succ 1$.

If T is real, we have

$$\int_2^{2+iT} \frac{\zeta'(s)}{\zeta(s)} \, ds = \log\zeta(2+iT) - \log\zeta(2)$$

and

$$\int_2^{2+iT} \frac{L'(s)}{L(s)} \, ds = \log L(2+iT) - \log L(2).$$

Here

$$|\zeta(2+iT) - 1| = \left|\sum_{n=2}^{\infty} n^{-2-it}\right| \leq \sum_{n=2}^{\infty} \left|n^{-s}\right| = \zeta(2) - 1 = 0.644934$$

and

$$|L(2+iT) - 1| = \left|\sum_{n=1}^{\infty} (-1)^n (2n+1)^{-2-it}\right| < \left|\sum_{n=2}^{\infty} (2n-1)^{-2-it}\right|$$

$$< \left|\left(1 - 2^{-s}\right)\right| \sum_{n=2}^{\infty} \left|n^{-s}\right| = \frac{3}{4}(\zeta(2) - 1) = 0.4837$$

Since these are less than 1, $\zeta(2+iT)$ and $L(2+iT)$ are constrained to a circle which excludes the origin, and

$$|\zeta(2+iT)| > 1 - 0.644934 \quad \text{and} \quad |L(2+iT)| > 1 - 0.4837 \qquad (6.3)$$

Finally, we have that

$$\int_2^{2+iT} \frac{\zeta'(s)}{\zeta(s)} \, ds = O(1) \quad \text{and} \quad \int_2^{2+iT} \frac{L'(s)}{L(s)} \, ds = O(1) \qquad (6.4)$$

To complete the proof of (6.1) we show that

$$\Im\left(\int_{2+iT}^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)} \, ds + \int_{2+iT}^{\frac{1}{2}+iT} \frac{L'(s)}{L(s)} \, ds\right) = O(\log T).$$

We assume that the path from $2 + iT$ to $\frac{1}{2} + iT$ does not pass through any zero of $\zeta(s)$ and any zero of $L(s)$, by moving the path up slightly if necessary.

By the Argument Principle the two integrals represent respectively the change in the argument of $\zeta(s)$ and the change in the argument of $L(s)$ as s moves from $2 + iT$ to $\frac{1}{2} + iT$.

These are approximately $\pi(c_1 + c_2)$, where c_1 is the number of sign changes in $\Re\zeta(s + it)$ and c_2 is the number of sign changes in $\Re L(s + it)$, as s moves from 2 to $\frac{1}{2}$, since the sign must change every time the argument changes by π.

We note that if s is real:

$$\Re\zeta(s + it) = \frac{1}{2}\left[\zeta(s + it) + \zeta(s - it)\right]$$

and

$$\Re L(s + it) = \frac{1}{2}\left[L(s + it) + L(s - it)\right].$$

Therefore, it is sufficient to show that the number of zeros of $\frac{1}{2}[\zeta(s+it)+\zeta(s-it)]$ and the number of zeros of $\frac{1}{2}[L(s+it)+L(s-it)]$ on the segment $\left[\frac{1}{2}, 2\right]$ of real axis are $O(\log T)$. In fact, we will use Proposition 6.2 to estimate the number of zeros of $f(s) = \frac{1}{2}\left[\zeta(s + it) + \zeta(s - it)\right]$ and the number of zeros of $g(s) = \frac{1}{2}\left[L(s + it) + L(s - it)\right]$ inside the circle $|s - 2| \prec \frac{3}{2}$.

We take $a = 2$, $R = \frac{7}{4}$ and $r = \frac{3}{2}$ in the Proposition 6.2.

First, we note that $|f(2)|$ and $|g(2)|$ are bounded by (6.3).

On the other hand,

$$\max_{|s-2|=7/4} |f(s)| = O\left(T^{3/4}\right) \quad \text{and} \quad \max_{|s-2|=7/4} |g(s)| = O\left(T^{3/4}\right)$$

by Proposition 6.3. Therefore if n is the number of zeros of $f(s)$ inside $|s-2| < \frac{3}{2}$ and if m is the number of zeros of $g(s)$ inside $|s - 2| < \frac{3}{2}$, we have

$$\left(\frac{7/4}{3/2}\right)^n = O\left(T^{3/4}\right) \quad \text{and} \quad \left(\frac{7/4}{3/2}\right)^m = O\left(T^{3/4}\right),$$

or taking logarithms in the first case we have that $n \log(7/6)$ is bounded by $\frac{3}{4}\log(T)$ plus a constant and the latter case we have that $m \log(7/6)$ is bounded by $\frac{3}{4}\log(T)$ plus a constant.

This completes the proof of (iii). $\square$

7 Conclusion

The symmetric form of the functional equation for $\zeta(s)$ represents one of the most remarkable achievements by B. Riemann.

This fundamental result was discovered and proved in his paper [15] in two different ways: the first was described in Section 2, the latter is similar to the one that we have illustrated in Section 3: it is conceptually more difficult because required taking the Mellin transform to boot and use an integral involving the theta function.

All modern proofs of the functional equation involve mathematical tools that were unavailable to L. Euler and it is remarkable that he was nevertheless able to predict the asymmetric form of the functional equation for the Zeta function.

In his paper [8] Euler used the differentiation of divergent series and a version of his of the Euler–Maclaurin summation formula.

Here we presented a proof of symmetric form of the functional equation for the Zeta function that Euler himself could have proved with a little step at end of his paper.

The result of this simple proof, based upon the three reflection formulae of $\eta(s)$, $L(s)$ and $\Gamma(s)$ with the duplication formula of *sine*, is a most general form of the functional equation for Riemann Zeta function.

It is easy to see that if $f(s)$ and $g(s)$ are two Dirichlet series, each satisfying a functional equation, then the product $f(s) \cdot g(s)$ defines a third Dirichlet series also satisfying a given functional equation, but, in our specific case, with the product of two functional equation in the asymmetric form we have obtained a functional equation in the unexpected symmetric form.

In the first part of this paper we obtained also an amazing connection between the Jacobi's imaginary transformation and an infinite series identity of Ramanujan.

In the second part using techniques similar to those of Riemann, it is shown how to locate and count the imaginary zeros of the entire function $\Im(s)$, which is an extension of the special function $A(s)$, that we have previously introduced [14].

Here we apply also the Euler–Boole summation formula and we obtain an estimate of the distribution of the zeros of the function $\Im(s)$ to follow a method, which Ingham [10, pp. 68–71] attributes to Backlund [1].

Basically we use the fact that we have a bound on the growth of $\zeta(s)$ and the growth of $L(s)$ in the critical strip.

More precisely with the Fundamental Theorem we also established that the number of the zeros of the function $\Im(s)$ in the critical strip is:

$$N_\Im(T) = \frac{T}{\pi} \log \frac{2T}{\pi e} + O(\log T) \tag{7.1}$$

Now, from Appendix, we have that the number of zeros of the factor

$$h(s) = \left(1 - 2^s\right)\left(1 - 2^{1-s}\right)$$

is:

$$N_h(T) = \frac{T}{\pi} \log 2 + O(1). \tag{7.2}$$

Subtracting (7.2) from (7.1) we have the number of zeros of the special function $A(s)$, that is:

$$N_A(T) = \frac{T}{\pi} \log \frac{T}{\pi e} + O(\log T). \tag{7.3}$$

and from [18, p. 214, 9.4.3] we have that the distribution function for the zeros of the Riemann Zeta function is:

$$N_\zeta(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{7.4}$$

Now subtracting (7.4) from(7.3) we have the number of zeros of the Dirichlet L function:

$$N_L(T) = \frac{T}{2\pi} \log \frac{2T}{\pi e} + O(\log T). \tag{7.5}$$

The previous results describe, in detail, the structure of the complex roots of the entire function $\Im(s)$.

Table 1 shows the frequency distribution for the actual zeros in successive intervals of t.

t	N_h	N_ζ	N_L	N_A	$N_\Im$
0–10	2	0	1	1	3
10–20	2	1	4	5	7
20–30	2	2	5	7	9
30–40	2	3	4	7	9
40–50	2	4	6	10	12
50–60	2	3	5	8	10
60–70	2	4	6	10	12
70–80	2	4	6	10	12
80–90	2	4	7	11	13
90–100	4	4	6	10	14
0–100	22	29	50	79	101
Eq. (7.2), (7.4), (7.5)	22	28	50		
Eq. (7.3), (7.1)				78	100

Table 1 Number of zeros of $h(s)$, $\zeta(s)$, $L(s)$, $A(s)$ and $\Im(s)$ in successive intervals of t

The author used M. Rubinstein's L-function calculator[4] to compute, with approximation, the complex zeros in the critical line $\sigma = 1/2$ and in the interval $0 \leq t \leq 100$ (see Fig. 2).

This makes the strong difference in the distributions of the gaps, all very interesting.

In this case, having also $N_A \geq N_A(T = 100)$, it follows that there are exactly N zeros in this portion of the critical strip, all lying on the critical line.

To be complete, we give also, for large T, the following result:

$$N_L(T) = N_\zeta(T) + N_h(T) \quad \text{and} \quad N_\Im(T) = 2\,N_L(T).$$

In addition we observe that the complex roots of the factor $h(s)$ lie on the vertical lines $\Re(s) = 0$ and $\Re(s) = 1$ and they are separated by $\frac{2\pi i}{\log 2}$.

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[4] http://oto.math.uwaterloo.ca/∼mrubinst/L_function_public/L.html.

$\zeta(1/2 + it)$	$L(1/2 + it)$	
14.13472514173469379046	6.02094890469759665490	67.63692086354606839805
21.02203963877155499263	10.24377030416655455214	68.36588450383442296123
25.01085758014568876321	12.98809801231242250745	70.18587990880211206137
30.42487612585951321031	16.34260710458722219498	72.15548497438188121469
32.93506158773918969066	18.29199319612353483853	73.76763552148589333615
37.58617815882567125722	21.45061134398346049720	75.14312164743311140580
40.91871901214749518740	23.27837652045953153182	76.69630320343019906457
43.32707328091499951950	25.72875642508872756727	78.80999831432091300369
48.00515088116715972794	28.35963434302532778565	80.21013123836663891515
49.77383247767230218192	29.65638401459315272181	81.21395162688315115773
52.97032147771446064415	32.59218652711715513082	83.66665601447057165128
56.44624769706339480437	34.19995750921314691304	84.73174036378162860822
59.34704400260235307965	36.14288045830313783057	86.57766016839026441021
60.83177852460980984426	38.51192314171869129378	87.62971811958789968904
65.11254404808160666088	40.32267406669054418034	89.80113161669581132597
67.07981052949417371448	41.80708462000456233716	91.34970381469757347393
69.54640171117397925293	44.61789105866230339348	92.23749991045425804600
72.06715767448190758252	45.59958439679156674594	94.16661958596002130705
75.70469069908393316833	47.74156228093914125078	96.13601116178055818527
77.14484006887480537268	49.72312923782586066576	96.96174157941748357761
79.33737502024936792276	51.68609345287052843953	98.75530041575452766860
82.91038085408603018316	52.76882076780472926504	
84.73549298051705010574	55.26754358469922484672	
87.42527461312522940653	56.93437405520229688680	
88.80911120763446542368	58.11670711067391797726	
92.49189927055848429626	60.42171394900783467302	
94.65134404051988696660	62.00863228576776945193	
95.87063422824530975874	63.71464111878543312352	
98.83119421819369223332	64.97617057309599934861	

Figure 2 Tables with approximate values of $t \in [0, 100]$ of the zeros of Riemann's Zeta function and of Dirichlet's Beta function on the critical line

While if we assume the Generalized Riemann Hypothesis (GRH)[5], this implies that all complex zeros of the special function $A(s)$ lie on the vertical line $\Re(s) = \frac{1}{2}$ and thus, at a height T the average spacing between zeros is asymptotic to $\frac{\pi}{\log T}$.

8   Appendix

We study the solution in s of the following Dirichlet polynomial:

$$f(s) = 1 - 2^{1-s} = 1 - 2\left(\frac{1}{2}\right)^s = 0. \tag{8.1}$$

This is the simplest example of a Dirichlet polynomial equation.
In this case, the complex roots are

$$s = 1 \pm \frac{2\pi i k}{\log 2} \quad \text{with } k \in Z.$$

Hence the complex roots lie on the vertical line $\Re(s) = 1$ and are separated by $\frac{2\pi i}{\log 2}$.

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[5] GRH: Riemann Hypothesis is true and in addition the nontrivial zeros of all Dirichlet L-functions lie on the critical line $\Re(s) = 1/2$.

In order to establish the density estimate of the roots of (8.1), we will estimate the winding number of the function $f(s) = 1 - 2\left(\frac{1}{2}\right)^s$ when s runs around the contour $C_1 + C_2 + C_3 + C_4$, where C_1 and C_3 are the vertical line segments $2 - iT \to 2 + iT$ and $-1 + iT \to -1 - iT$ and C_2 and C_4 are the horizontal line segments $2 + iT \to -1 + iT$ and $-1 - iT \to 2 - iT$, with $T > 0$ (see Fig. 1).

For $\Re(s) = 2$ we have $|1 - f(s)| = \left|2\left(\frac{1}{2}\right)^s\right| = \frac{1}{2} < 1$, so the winding number along C_1 is at most $\frac{1}{2}$. Likewise, for $\Re(s) = -1$, we have

$$1 < |f(s) - 1| = \left|2\left(\frac{1}{2}\right)^{-1+iT}\right| \le 2\left(\frac{1}{2}\right)^{-1} = 4$$

so the winding number along C_3 is that of term $2\left(\frac{1}{2}\right)^s$, up to at most $\frac{1}{2}$.

Hence, the winding number along the contour $C_1 + C_3$ is equal to $\left(\frac{T}{\pi}\right)\log 2$, up to at most 1.

We will now show that the winding number along $C_2 + C_4$ is bounded, using a classical argument [10, p. 69].

Let n the number of distinct points on C_2 at which $\Re f(s) = 0$.

For real value of z,

$$\Re f(z + iT) = \frac{1}{2}\left[f(z + iT) + f(z - iT)\right].$$

Hence, putting $g(z) = 2\Re f(z + iT)$ we see that n is bounded by the number of zeros of g in a disk containing the interval $(0, 1)$.

We take the disk centred at 2, with radius 3. We have

$$|g(2)| \ge 2 - 2 \cdot 2\left(\frac{1}{2}\right)^2 = 1 > 0.$$

Furthermore, let G the maximum of g on disk with the same centre and radius $e \cdot (3)$, so

$$G \le 2 + 2 \cdot \left(\frac{1}{2}\right)^{2 - e \cdot 3}.$$

By Proposition 6.2, it follows that $n \le \log|G/g(2)|$.

This gives a uniform bound on the winding number over C_2. The winding number over C_4 is estimated in the same manner.

We conclude from the above discussion that the winding number of $f(s) = 1 - 2^{1-s}$ over the closed contour $C_1 + C_2 + C_3 + C_4$ equals $\left(\frac{T}{\pi}\right)\log 2$, up to a constant (dependent on f), from which follows the asymptotic density estimate:

$$D_f = \left(\frac{T}{\pi}\right)\log 2 + O(1).$$

If we count the zeros in the upper half of a vertical strip $\{s \colon 0 \le \Im(s) \le T\}$ we have:

$$N_f = \left(\frac{T}{2\pi}\right)\log 2 + O(1).$$

A lot of details in relation to what we have just shown were published in [12, chap. 3, pp. 63–77].

9 Additional Remark

The author is aware that some of the results presented in [14] and in this paper are not new.

In particular, the main subject of this paper, the function $\Im(s)$ is, apart from a factor

$$\left(1 - 2^s\right)\left(1 - 2^{1-s}\right)\Gamma(s)/\pi^s,$$

equal to the product of the Riemann Zeta function and a certain L-function.

That product is equal to the Dedekind Zeta function associated to the algebraic number field obtained from the field of rational number by adjoining a square root of -1.

Let $r_2(n)$ denote the number of ways to write n as sum of two squares, then the generating series for $r_2(n)$:

$$\zeta_{Q(\sqrt{-1})}(s) = \frac{1}{4}\sum_{n=1}^{\infty} r_2(n)\,(n)^{-s}$$

is precisely the Dedekind Zeta function of the number field $Q(\sqrt{-1})$, because it counts the number of ideals of norm n.

It factors as the product of two Dirichlet series:

$$\zeta_{Q(\sqrt{-1})}(s) = \zeta(s)\,L(s, \chi_4).$$

The factorization is a result from class field theory, which reflects the fact that an odd prime can be expressed as the sum of two squares if and only if it is congruent to 1 modulo 4.

Dedekind Zeta functions were invented in the 19th century, and in the course of time many of their properties have been established. Some of the present results are therefore special cases of well-known properties of the Dedekind Zeta.

Nevertheless, the goal of this manuscript is to highlight some demonstration, direct and by increments, for treating certain functional equations and special functions involved, as inspired by methods similar to the ones used by Euler in his paper [8] and in many other occasions (see [20], [19], and [21, chap. 3]).

In order not to leave unsatisfied the reader's curiosity, we recall that the choice of the letter $\Im$ for the special function $\Im(s)$ is in honour of Эйлер (Euler).

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