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Stability, Boundedness and Existence of Periodic Solutions to Certain Third Order Nonlinear Differential Equations

A. T. ADEMOLA [1a], M. O. OGUNDIRAN [1b],
P. O. ARAWOMO [2],

[1] *Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria*
[a] *e-mail: ademola672000@yahoo.com*
[b] *e-mail: adeolu74113@yahoo.com*

[2] *Department of Mathematics, University of Ibadan, Ibadan, Nigeria*
e-mail: womopeter@gmail.com

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Abstract

In this paper, criteria are established for uniform stability, uniform ultimate boundedness and existence of periodic solutions for third order nonlinear ordinary differential equations. In the investigation Lyapunov's second method is used by constructing a complete Lyapunov function to obtain our results. The results obtained in this investigation complement and extend many existing results in the literature.

Key words: Third order, nonlinear differential equation, uniform stability, uniform ultimate boundedness, periodic solutions.

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1 Introduction

In order to describe and understand natural occurrence (or processes), we search for pattern of occurrences of these natural processes. Patterns that repeat are particularly useful and of interest, because we can predict their future behaviour. We meet with periodicity when something is repeated in time and in space. The sun rises everyday, the rotation of the planets, changes of seasons, high and low tides, predator and pre-population, data with periodic influences are examples of cyclic or periodic patterns. Ancient astronomers and astrologers used these observations to regulate the activities of individuals, communities, nations or countries. Researches in mathematical, physical, biological and social sciences about these processes requires information about the existence and nonexistence, stability and instability, boundedness and unboundedness of solutions of the derived related models or systems.

Many works have been done by notable authors on the stability and boundedness of solutions of nonlinear ordinary differential equations, see for instance LaSalle and Lefschets [28], Reissig et. al [31], Rouche et. al [32], Yoshizawa [42, 43] which contain general results on the subject matters, and the papers of Ademola et. al [1]–[10], Afuwape and Adesina [11], Andres [12], Antoisewicz [13], Bereketoğlu and Györi [15], Chukwu [17], Ezeilo [18, 19, 20], Hara [21], Mehri and Shadman [23] Ogundare [29], Omeike [30], Swick [34], Tejumola [35, 36], Tunç [38, 40] and the references cited therein.

Up to now, according to our investigation in the relevant literatures, the problem of existence of periodic solutions for various nonlinear second and third order ordinary differential equations have been discussed in the literature by few authors; see for example, the paper of Ezeilo [18], Mehri et. al [22, 24, 26], Minhós [27], Shadman and Mehri [33] and Tunç et. al [37, 39]. These works were done using topological degree theory or the Leray–Schauder principle. The purpose of this paper is to establish criteria for existence of periodic, uniform stability, and uniform ultimate boundedness of solutions for the third order nonlinear ordinary differential equations

$$\dddot{x} + \lambda\phi(t)g_1(x, \dot{x}, \ddot{x}, \sigma) = p(t, x, \dot{x}, \ddot{x}) \tag{1.1}$$

where g_1 are functions sufficiently smooth with respect to their arguments, ϕ and p are ω periodic functions and $\lambda > 0$ is a constant. Let $x(t, \sigma)$, as in [22], be a solution of (1.1), where σ is a parameter. If $\sigma = 0$ and we assume further that

$$g_1(x, \dot{x}, \ddot{x}, 0) = f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x}) + h(x),$$

then equation (1.1) becomes

$$\dddot{x} + \lambda\phi(t)[f(x, \dot{x}, \ddot{x})\ddot{x} + g(x, \dot{x}) + h(x)] = p(t, x, \dot{x}, \ddot{x}). \tag{1.2}$$

Setting $\dot{x} = y$ and $\ddot{x} = z$ Eq. (1.2) becomes

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = p(t, x, y, z) - \lambda\phi(t)[f(x, y, z)z + g(x, y) + h(x)] \tag{1.3}$$

where $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$, $f \in C(\mathbb{R}^3, \mathbb{R})$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R})$, $\phi \in C(\mathbb{R}^+, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. The derivatives $f_x(x, y, z)$, $f_z(x, y, z)$, $g_x(x, y)$, $h'(x)$ and $\phi'(t)$ exist and continuous for all t, x, y, z with $h(0) = 0$. The dots, as usual, stand for differentiation with respect to the independent variable t. Motivation for this work come from the works of Mehri and Niksirat [22], Minhós [27] and Tunç [39] where results on periodicity were proved. According to our observation from the relevant literature, this paper is one of the few articles on the existence of periodic solutions for third order nonlinear differential equations where a complete Laypunov function is used.

2 Preliminaries

Consider a system of differential equations

$$\frac{dX}{dt} = F(t, X) \tag{2.1}$$

where X is an n-vector. Suppose that $F(t, X)$ is continuous in (t, X) on $\mathbb{R}^+ \times \mathbb{D}$ where $\mathbb{D}$ is a connected open set in $\mathbb{R}^n$. Let C be a class of solutions of (2.1) which remain in $\mathbb{D}$ and let X_0 be an element of C. We have the following results:

Lemma 2.1 [42] *Suppose that there exists a Lyapunov function $V(t, X)$ defined on R^+, $\|X\| < H$ which satisfies the following conditions:*

(i) $V(t, 0) \equiv 0$*;*

(ii) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$*,* a*,* b *are continuous and increasing;*

(iii) $V_{(2.1)}(t, X) \leq -c(\|X\|)$ *for all* $(t, X) \in \mathbb{R}^+ \times D$*.*

Then the trivial solution $X(t) \equiv 0$ of (2.1) is uniformly asymptotically stable.

Lemma 2.2 [41, 42] *Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+$, $\|X\| \geq R$, where R may be large, which satisfies the following conditions:*

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$*, where* $a(r)$ *and* $b(r)$ *are continuous and increasing and* $a(r) \to \infty$ *as* $r \to \infty$*;*

(ii) $V'_{(2.1)}(t, X) \leq -c(\|X\|)$*, where* $c(r)$ *is positive and continuous,*

then the solution of (2.1) are uniformly ultimately bounded.

Lemma 2.3 [41, 42] *If there exists a Lyapunov function satisfying the condition of Lemma 2.2, then (2.1) has at least a periodic solution of period ω.*

3 Main Results

Assumptions In addition to the basic conditions on the functions f, g, h, p and ϕ, suppose that $a, b, c, a_1, b_1, \delta, \mu_0, \mu_1$ and ϵ_0 are positive constants and that for all $t \geq 0$;

(i) $\mu_0 \leq \phi(t) \leq \mu_1$, $\max_{t \in \mathbb{R}^+} |\phi'(t)| \leq \epsilon_0$;

(ii) $a \leq f(x, y, z) \leq a_1$ for all x, y, z and $y f_x(x, y, 0) \leq 0$ for all x, y;

(iii) $b \leq \frac{g(x,y)}{y} \leq b_1$ for all $x, y \neq 0$ and $g_x(x, y) \leq 0$ for all x, y;

(iv) $h(0) = 0$, $\frac{h(x)}{x} \geq \delta$ for all $x \neq 0$;

(v) $h'(x) \leq c$ for all x and $ab - c > 0$;

(vi) $|p(t,x,y,z)| \leq \varphi(t) + \psi(t)(|x|+|y|+|z|)$ where φ and ψ are periodic functions of t satisfying

$$\varphi(t) \leq M \tag{3.1a}$$

$0 < M < \infty$ and there exists $\epsilon_1 > 0$ such that

$$0 \leq \psi(t) \leq \epsilon_1 \tag{3.1b}$$

When $p(t,x,y,z) \equiv 0$ (1.3) comes to be

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -\lambda\phi(t)[f(x,y,z)z + g(x,y) + h(x)] \tag{3.2}$$

where f, g and h are the functions defined in Section 1. We have the following results.

Theorem 3.1 *If assumptions (i)–(v) hold true, then the trivial solutions of* (3.2) *is uniformly asymptotically stable.*

Theorem 3.2 *If assumptions (i)–(vi) hold, then the solution of* $(x(t),\ y(t),\ z(t))$ *of* (1.3) *is uniformly ultimately bounded.*

Theorem 3.3 *If the assumptions of Theorem 3.2 hold, then* (1.3) *has a periodic solution of period* ω.

Corollary 3.4 *If assumptions (i)–(v) hold and (vi) is replaced by* $p(t)$ $p \in C(\mathbb{R}^+, \mathbb{R})$ *periodic in* t *of period* ω*, and bounded above by a finite constant,*

(i) then the solution $(x(t), y(t), z(t))$ *of* (1.3) *is uniformly ultimately bounded.*

(ii) If Corollary 3.4 (i) holds good, then (1.3) *has a periodic solution of period* ω.

Remark 3.5

(i) Whenever $\lambda\phi(t)f(x,y,z) = a$, $\lambda\phi(t)g(x,y) = g(y)$, $\lambda\phi(t)h(x) = cx$ and $p(t,x,y,z) = p(t)$, equation (1.3) reduces to the case discussed by Minhós in [27].

(ii) If $\lambda\phi(t)f(x,y,z) = c_2(t)$ and $\lambda\phi(t)g(x,y) = c_1(t)y$, (1.3) specializes to that studied by Tunç in [39].

(iii) In the case $\lambda\phi(t)f(x,y,z) = \psi(y)$, $\lambda\phi(t)g(x,y) = k^2 + \phi(x)$, $p(t,x,y,z) = e(t)$, system (1.3) reduces to that discussed by Mehri and Shadman in [24, 25].

(iv) If $p(t,x,y,z) = 0$ equation (1.1) specializes to (3.1) discussed in [22]. Also, if $\lambda\phi(t)f(x,y,z) = f(y)h(z)$ and $g(x,y) = 0 = h(x)$, equation (1.3) specializes to (3.3) discussed also in [22].

(v) In [22, 25, 26, 39] Leray–Schauder approach was used by these authors to establish existence of periodic solutions to the third order ordinary differential equations considered. In this paper Lyapunov's second method is used, by constructing a complete Lyapunov function, to obtain criteria for existence of periodic solutions. Thus, our assumptions are completely different.

The main tool used in this investigation is the continuously differentiable function

$$
\begin{aligned}
2V = {} & 2[\alpha_1 + a\lambda\phi(t)]\lambda\phi(t)\int_0^x h(\xi)d\xi + 4\lambda\phi(t)yh(x) + 2a\alpha_2\lambda\phi(t)xy \\
& + 4\lambda\phi(t)\int_0^y g(x,\tau)d\tau + 2[\alpha_1 + a\lambda\phi(t)]\lambda\phi(t)\int_0^y \tau f(x,\tau,0)d\tau + 2z^2 \\
& + \alpha_2 y^2 + \alpha_2 b\lambda\phi(t)x^2 + 2\alpha_2 xz + 2[\alpha_1 + a\lambda\phi(t)]yz
\end{aligned}
\tag{3.3}
$$

where α_1 and α_2 are positive constants satisfying

$$c < \alpha_1 b < ab\lambda\mu_0 \tag{3.4a}$$

and

$$\alpha_2 < \min\left\{b\lambda\mu_0, \lambda\mu_0(ab\lambda\mu_0 - c)A_0, \frac{1}{2}(a\lambda\mu_0 - \alpha_1)A_1\right\} \tag{3.4b}$$

where

$$A_0 = \left[1 + a\lambda\mu_0 + \lambda\mu_0\delta^{-1}\left(\frac{g(x,y)}{y} - b\right)^2\right]^{-1}$$

and

$$A_1 = \left[1 + \lambda\mu_0\delta^{-1}\big(f(x,y,z) - a\big)^2\right]^{-1}.$$

Remark 3.6 The differentiable function $V = V(t,x,y,z) = V(t,x(t),y(t),z(t))$ defined in (3.3) is similar to the one used in [7].

Lemma 3.7 Subject to assumptions (i)–(v), $V(t,0,0,0) = 0$; and there exist positive constants $D_0 = D_0(a,b,c,\alpha_1,\alpha_2,\delta,\lambda,\mu_0)$ and $D_1 = D_1(a,b,c,\alpha_1,\alpha_2,a_1,b_1,\lambda,\mu_1)$ such that

$$D_0(x^2(t) + y^2(t) + z^2(t)) \le V(t,x,y,z) \le D_1(x^2(t) + y^2(t) + z^2(t)), \tag{3.5a}$$

$$V(t,x,y,z) \to +\infty \text{ as } x^2(t) + y^2(t) + z^2(t) \to \infty, \tag{3.5b}$$

also, there exist positive constants $D_2 = D_2(a,b,c,\alpha_1,\alpha_2,\delta,\epsilon_0,\lambda,\mu_0)$ and $D_3 = D_3(a,\alpha_1,\ \alpha_2,\lambda,\mu_0)$ such that along any solution $(x(t),y(t),z(t))$ of (1.3)

$$\dot{V} = \frac{d}{dt}V(t,x,y,z) \le -D_2(x^2(t) + y^2(t) + z^2(t)) + D_3(|x| + |y| + |z|)|p(t,x,y,z)|. \tag{3.5c}$$

Proof Obviously, $V(t,0,0,0)=0$ for all $t\in\mathbb{R}^+$. Since $h(0)=0$ the function V defined in (3.3) can be rearranged in the form

$$\begin{aligned}V = b^{-1}\lambda\phi(t)\int_0^x \Big[\alpha_1 b - h'(\xi) + ab\lambda\phi(t) - h'(\xi)\Big]h(\xi)d\xi + \alpha_2 y^2 \\ + \frac{1}{2}\Big(\alpha_1 y + z\Big)^2 + b\lambda\phi(t)\Big(y + b^{-1}h(x)\Big)^2 \\ + \frac{1}{2}\Big(\alpha_2 x + a\lambda\phi(t)y + z\Big)^2 + 2\lambda\phi(t)\int_0^y\Big(\frac{g(x,\tau)}{\tau} - b\Big)\tau d\tau \\ + \int_0^y\Big\{\alpha_1\Big[\lambda\phi(t)f(x,\tau,0) - \alpha_1\Big] + a\lambda^2\phi^2(t)[f(x,\tau,0)-a]\Big\}\tau\, d\tau \\ + \frac{\alpha_2}{2}\Big(b\lambda\phi(t) - \alpha_2\Big)x^2\end{aligned}$$

Now, since $\phi(t)\geq\mu_0$ for all $t\geq 0$, $f(x,y,z)\geq a$ for all x,y,z, $g(x,y)\geq by$ for all x and $y\neq 0$, $h(x)\geq\delta x$ for all $x\neq 0$, and $h'(x)\leq c$ for all x, we have

$$\begin{aligned}V \geq \frac{1}{2}\Big[b^{-1}\delta\lambda\mu_0(\alpha_1 b - c + ab\lambda\mu_0 - c) + \alpha_2(b\lambda\mu_0 - \alpha_2)\Big]x^2 \\ + b^{-1}\lambda\mu_0\Big(\delta x + by\Big)^2\frac{1}{2}\Big[\alpha_1(a\lambda\mu_0 - \alpha_1) + 2\alpha_2\Big]y^2 \\ + \frac{1}{2}\Big(\alpha_1 y + z\Big)^2 + \frac{1}{2}\Big(\alpha_2 x + a\lambda\mu_0 y + z\Big)^2.\end{aligned}$$

From (3.4a), (3.4b), $\alpha_1 b > c$, $ab\lambda\mu_0 > c$, $a\lambda\mu_0 > \alpha_1$ and $b\lambda\mu_0 > \alpha_2$, hence, there exist positive constants δ_0, K_0, K_1, K_2 such that

$$V \geq \delta_0(x^2 + y^2 + z^2) \tag{3.6}$$

for all $t\in\mathbb{R}^+$ and $(x,y,z)\in\mathbb{R}^3$, where $\delta_0 = \min\{K_0, K_1, K_2\}$,

$$\begin{aligned}K_0 = \frac{\lambda\mu_0}{b}\min\{\delta, b\} + \frac{\delta\lambda\mu_0}{2b}(\alpha_1 b - c + ab\lambda\mu_0 - c) + \alpha_2(b\lambda\mu_0 - \alpha_2) \\ + \frac{1}{2}\min\{\alpha_1, a\lambda\mu_0, 1\},\end{aligned}$$

$$K_1 = \frac{\alpha_1}{2}(a\lambda\mu_0 - \alpha_1) + \alpha_2 + \frac{1}{2}\min\{\alpha_1, 1\} + \frac{\lambda\mu_0}{b}\min\{\delta, b\} + \frac{1}{2}\min\{\alpha_1, a\lambda\mu_0, 1\}$$

and

$$K_2 = \frac{1}{2}[\min\{\alpha_1, 1\} + \min\{\alpha_1, a\lambda\mu_0, 1\}].$$

Furthermore, since $\phi(t)\leq\mu_1$, $0\leq t<\infty$, $f(x,y,z)\leq a_1$ for all x,y and z, $g(x,y)\leq b_1$ for all x and $y\neq 0$, $h'(x)\leq c$ for all x, and the fact that $h(0)=0$, (3.3) becomes

$$V \leq \frac{1}{2}\big[(\alpha_1 + a\lambda\mu_1)c\lambda\mu_1 + \alpha_2 b\lambda\mu_1\big]x^2$$
$$+ \left[\frac{1}{2}a_1\lambda\mu_1(\alpha_1 + a\lambda\mu_1) + \frac{1}{2}\alpha_2 + b_1\lambda\mu_1\right]y^2 + z^2$$
$$+ 2(c\lambda\mu_1 + \frac{1}{2}a\alpha_2\lambda\mu_1)xy + \alpha_2 xz + (\alpha_1 + a\lambda\mu_1)yz.$$

Recall that $2mn \leq m^2+n^2$, hence, there exist positive constants δ_1, K_3, K_4 and K_5 such that

$$V \leq \delta_1(x^2 + y^2 + z^2) \tag{3.7}$$

for all $t \in \mathbb{R}^+$ and $(x, y, z) \in \mathbb{R}^3$, where $\delta_1 = \max\{K_3, K_4, K_5\}$, $K_3 = \frac{1}{2}[\alpha_2((a + b)\lambda\mu_1 + 1) + c\lambda\mu_1(\alpha_1 + a\lambda\mu_1 + 2)]$, $K_4 = \frac{1}{2}[\alpha_1 + \alpha_2 + [a_1(\alpha_1 + a\lambda\mu_1) + 2b_1 + a(1 + \alpha_2) + 2c]\lambda\mu_1]$ and $K_5 = \frac{1}{2}(\alpha_1 + \alpha_2 + a\lambda\mu_1)$.
Moreover, by estimate (3.6), we have $V(t, 0, 0, 0) = 0$ if and only if $x^2+y^2+z^2 = 0$, $V(t, x, y, z) > 0$ if and only if $x^2 + y^2 + z^2 \neq 0$, it follows that

$$V(t, x, y, z) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{3.8}$$

Hence, from (3.6), (3.7) and (3.8) estimates (3.5a) and (3.5b) are satisfied with $\delta_0 \equiv D_0$ and $\delta_1 \equiv D_1$ in (3.6) and (3.7) respectively.

Besides, along any solution $(x(t), y(t), z(t))$ of (1.3)

$$\dot{V}_{(1.3)} = a\alpha_2\lambda\phi(t)y^2 + 2\alpha_2 yz + [\alpha_2 x + (\alpha_1 + a\lambda\phi(t))y + 2z]p(t, x, y, z)$$
$$+ \sum_{i=1}^{2} W_i - \sum_{i=3}^{4} W_i - \alpha_2\lambda\phi(t)\left[\frac{g(x, y)}{y} - b\right]xy - \alpha_2\lambda\phi(t)\big[f(x, y, z) - a\big]xz \tag{3.9}$$

where

$$W_1 = \lambda\phi'(t)\Bigg[(\alpha_1 + 2a\lambda\phi(t))\int_0^x h(\xi)\,d\xi + 2\int_0^y g(x,\tau)\,d\tau + 2yh(x)$$
$$+ (\alpha_1 + 2a\lambda\phi(t))\int_0^y \tau f(x,\tau,0)\,d\tau + \frac{1}{2}(b\alpha_2 x^2 + 2a\alpha_2 xy + 2ayz)\Bigg];$$
$$W_2 = \lambda\phi(t)\left[2y\int_0^y g_x(x,\tau)d\tau + (\alpha_1 + a\lambda\phi(t))y\int_0^y \tau f_x(x,\tau,0)\,d\tau\right];$$
$$W_3 = \alpha_2\lambda\phi(t)xh(x) + \lambda\phi(t)[(\alpha_1 + a\lambda\phi(t))yg(x,y) - 2y^2h'(x)]$$
$$+ [2\lambda\phi(t)f(x,y,z) - (\alpha_1 + a\lambda\phi(t))]z^2;$$
$$W_4 = (\alpha_1 + a\lambda\phi(t))\lambda\phi(t)yz[f(x,y,z) - f(x,y,0)].$$

Now, for all $t \in \mathbb{R}^+$ $\phi(t) \leq \mu_1$, $h(0) = 0$, $h'(x) \leq c$ for all x, $g(x, y) \leq b_1 y$ for all x and $y \neq 0$ and $f(x, y, z) \leq a_1$ for all x, y, z, it follows that

$$W_1 \leq K_7\phi'(t)(x^2 + y^2 + z^2)$$

for all x, y, z where

$$K_7 = \max\left\{\frac{\lambda}{2}[(\alpha_1 + 2a\lambda\mu_1 + 2)c + \alpha_2 b] + a\alpha_2, \right.$$
$$\left. \frac{\lambda}{2}[a_1(\alpha_1 + 2a\lambda\mu_1) + 2(b_1 + c)] + a(\alpha_2 + 1), a\right\}.$$

It is clear from conditions (ii) and (iii) that $yf_x(x, y, 0) \leq 0$ for all x, y and $g_x(x, y) \leq 0$ for all x and y, so that

$$W_2 \leq 0$$

for all x, y, z. Furthermore, $h(x) \geq \delta x$ for all $x \neq 0$, $g(x, y) \geq by$ for all x and $y \neq 0$, $h'(x) \leq c$ for all x and $f(x, y, z) \geq a$ for all x, y, z, from these inequalities we obtain

$$W_3 \geq \alpha_2\lambda\mu_0\delta x^2 + \lambda\mu_0(\alpha_1 b - c + ab\lambda\mu_0 - c)y^2 + (a\lambda\mu_0 - \alpha_1)z^2$$

for all x, y and z. Finally, since $yf_z(x, y, z) \geq 0$ for all x, y, z it follows that

$$W_4 = (\alpha_1 + a\lambda\phi(t))\lambda\phi(t)yz^2 f_z(x, y, \theta z) \geq 0$$

$0 \leq \theta \leq 1$ and $(\alpha_1 + a\lambda\phi(t))\lambda\phi(t)yz^2 f_z(x, y, \theta z) = 0$ when $z = 0$. Combining all estimates for W_i $(i = 1, 2, 3, 4)$ into (3.9), we obtain

$$\dot{V}_{(1.3)} \leq K_8(|x| + |y| + |z|)|p(t, x, y, z)| + K_7\phi'(t)(x^2 + y^2 + z^2) - \frac{1}{2}\alpha_2\lambda\mu_0\delta x^2$$
$$- [\lambda\mu_0(\alpha_1 b - c + ab\lambda\mu_0 - c) - \alpha_2(a\lambda\mu_0 + 1)]y^2 - [a\lambda\mu_0 - \alpha_1 - \alpha_2]z^2 - \sum_{i=5}^{6} W_i \tag{3.10}$$

where $K_8 = \max\{\alpha_2, \alpha_1 + a\lambda\mu_1, 2\}$, $W_5 = \frac{1}{4}\alpha_2\lambda\mu_0\delta x^2 + \alpha_2\lambda\mu_0[g(x, y) - by]x$ and $W_6 = \frac{1}{4}\alpha_2\lambda\mu_0\delta x^2 + \alpha_2\lambda\mu_0[f(x, y, z) - a]xz$. Since α_2, λ, μ_0 and δ are positive constants it is not difficult to show that

$$W_5 \geq -\alpha_2\delta^{-1}\lambda\mu_0\left[\frac{g(x, y)}{y} - b\right]^2 y^2 \quad \text{and} \quad W_6 \geq -\alpha_2\delta^{-1}\lambda\mu_0\big[f(x, y, z) - a\big]^2 z^2$$

for all x, y and z. Using estimates W_5 and W_6 in (3.10) and applying the inequality in (3.4b), there exist positive constants K_9, K_{10} such that

$$\dot{V}_{(1.3)} \leq -K_{10}(x^2 + y^2 + z^2) + K_8(|x| + |y| + |z|)|p(t, x, y, z)| \tag{3.11}$$

for all x, y, z where $K_{10} = K_9 - \epsilon_0 > 0$ and $K_9 = \min\{\frac{1}{2}\alpha_2\lambda\mu_0\delta, \lambda\mu_0(\alpha_1 b - c), \frac{1}{2}(a\lambda\mu_0 - \alpha_1)\}$. This completes the proof of Lemma 3.7. □

Proof of Theorem 3.1 Let $(x(t), y(t), z(t))$ be any solution of (3.2), By (3.3) it is clear that $V(t, 0, 0, 0) = 0$ for all $t \in \mathbb{R}^+$ and in view of estimates (3.6), (3.7), (3.8) and (3.11) when $p(t,x,y,z)=0$, all the hypotheses of Lemma 2.1 are satisfied, hence, by Lemma 2.1 the trivial solution of (3.2) is uniformly asymptotically stable. This completes the proof of Theorem 3.1. $\square$

Proof of Theorem 3.2 Suppose that $(x(t),y(t),z(t))$ be any solution of (1.3). From (3.11) and assumption (vi), we have

$$\dot{V}_{(1.3)} \le -K_{10}(x^2+y^2+z^2)+3^{1/2}K_8(x^2+y^2+z^2)^{1/2}\varphi(t)+3K_8(x^2+y^2+z^2)\psi(t)$$

for all $t\in\mathbb{R}^+$ and $(x,y,z)\in\mathbb{R}^3$. In view of the estimates (3.1a) and (3.1b), there exist positive constants K_{11} and K_{12} such that

$$\dot{V}_{(1.3)} \le -K_{11}(x^2+y^2+z^2)+K_{12}(x^2+y^2+z^2)^{1/2}$$

for all $t\ge 0$, x,y,z, where $K_{11}=K_{10}-3K_8\epsilon_1>0$ since ϵ_1 is chosen sufficiently small and $K_{12}=3^{1/2}K_8M>0$. Now choose $(x^2+y^2+z^2)^{1/2}\ge K_{13}$, there exists a constant $\delta_2>0$ such that

$$\dot{V}_{(1.3)} \le -\delta_2(x^2+y^2+z^2) \tag{3.12}$$

for all x,y,z and $t\ge 0$, where $K_{13}=2K_{11}^{-1}K_{12}>0$ may be large and for all $t\ge 0$, x,y,z, and $\delta_2=\frac{1}{2}K_{11}$. Thus, by (3.6), (3.7), (3.8) and (3.12) all hypotheses of Lemma 2.2 hold true. Hence, by Lemma 2.2, the solution $(x(t),y(t),z(t))$ of (1.3) is uniformly ultimately bounded. $\square$

Proof of Theorem 3.3 Let $(x(t),y(t),z(t))$ be any solution of (1.3). Since the function $V=V(t,x,y,z)$ defined in (3.3) satisfies the conditions of Theorem 3.2 so that the solutions of (3.3) are uniformly ultimately bounded and by Lemma 2.3 (1.2) and consequently (1.3) has at least a periodic solution of period ω. $\square$

Example 3.8 Consider the third order ordinary differential equation

$$\begin{aligned}\dddot{x}+\frac{1}{4}\left(1+\frac{1}{2+\sin 4t}\right)&\left[4\ddot{x}+\frac{\ddot{x}}{3+|x\dot{x}|+e^{u}}+2\dot{x}+\frac{\dot{x}}{4+|x\dot{x}|}+3x+\frac{x}{1+|x|}\right]\\ &=\left(1+\frac{\cos t}{2+\sin 4t}\right)+\big(1+\cos(t/2)\big)(|x|+|\dot{x}|+|\ddot{x}|),\end{aligned} \tag{3.13}$$

where

$$u=\frac{1}{4+|\dot{x}\ddot{x}|}.$$

Setting $\dot{x}=y$, $\ddot{x}=z$, (3.13) becomes

$$\begin{aligned}&\dot{x}=y,\quad \dot{y}=z,\quad \dot{z}=\left(1+\frac{\cos t}{2+\sin 4t}\right)+\big(1+\cos(t/2)\big)(|x|+|y|+|z|)\\ &-\frac{1}{4}\left(1+\frac{1}{2+\sin 4t}\right)\left[4z+\frac{z}{3+|xy|+e^{u_1}}+2y+\frac{y}{4+|xy|}+3x+\frac{x}{1+|x|}\right]\end{aligned} \tag{3.14}$$

where

$$u_1=\frac{1}{4+|yz|}.$$

From (1.3) and (3.14), we have the following relations:

(i) $\lambda = \frac{1}{4} > 0$ as required from the basic condition,

(ii) $\phi(t) = \left(1 + \dfrac{1}{2 + \sin 4t}\right) = \phi(t + \omega)$ for all $t \in \mathbb{R}^+$,

hence, ϕ is periodic in t with period ω (see Fig. 1). Also,

$$0 < \frac{1}{2 + \sin 4t} \leq 1 \quad \text{for all } t \in \mathbb{R}^+,$$

this implies that $1 \leq \phi(t) \leq 2$ for all t, where $1 = \mu_0 > 0$ and $2 = \mu_1 > 0$. Moreover,

$$\max_{t \in \mathbb{R}^+} |\phi'(t)| = \frac{4\cos 4t}{(2 + \sin 4t)^2} \leq 2 \quad \forall t \in \mathbb{R}^+,$$

where $2 = \epsilon_0 > 0$.

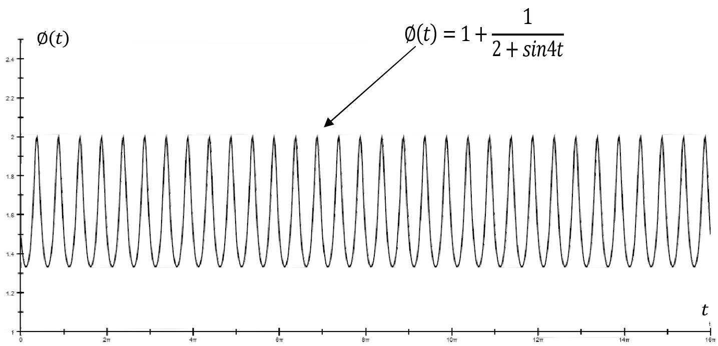

Fig. 1. Periodic function $\phi(t)$ for all $t \in \mathbb{R}^+$

(iii) $f(x, y, z) = 4 + \dfrac{1}{3 + |xy| + \exp u_1}$.

But

$$0 \leq \frac{1}{3 + |xy| + \exp u_1} \leq 1 \quad \forall x, y, z,$$

it follows that

$$4 \leq f(x, y, z) \leq 5 \quad \text{for all } x, y, z$$

where $4 = a > 0$ and $5 = a_1 > 0$.

Furthermore, for $x > 0$, we have

$$yf_x(x, y, z) = \frac{-y^2}{[3 + |xy| + \exp u_1]^2} \leq 0 \quad \text{for all } x, y, z.$$

Also, for $z > 0$, we have

$$yf_z(x, y, z) = \frac{y^2 \exp u_1}{[4 + |yz|][3 + |xy| + \exp u_1]^2} \geq 0 \quad \text{for all } x,\ y,\ z.$$

(iv) $g(x,y) = 2y + \dfrac{y}{4+|xy|}$.

But
$$0 < \frac{1}{4+|xy|} \le 1 \quad \text{for all } x \text{ and } y,$$
this implies that
$$2 \le \frac{g(x,y)}{y} \le 3 \quad \text{for all } x \text{ and } y \ne 0,$$
where $2 = b > 0$ and $3 = b_1 > 0$.

For $x > 0$
$$g_x(x,y) = \frac{-y^2}{[4+|xy|]^2} \le 0 \quad \text{for all } x, y$$

(v) $h(x) = 3x + \dfrac{x}{1+|x|}$,

it is clear from this relation that $h(0) = 0$. Also, since
$$\frac{1}{1+|x|} > 0 \quad \text{for all } x,$$
if follows that
$$\frac{h(x)}{x} \ge 3 \quad \text{for all } x \ne 0$$
where $3 = \delta > 0$. Moreover, for $x > 0$,
$$h'(x) = 3 + \frac{1}{(1+|x|)^2}.$$
Noting that
$$\frac{1}{(1+|x|)^2} \le 1 \quad \text{for all } x,$$
it follows that
$$h'(x) \le 4 \quad \text{for all } x$$
where $4 = c > 0$, and $ab > c$ implies that $2 > 1$.

(vi) $\varphi(t) = 1 + \dfrac{\cos t}{2+\sin 4t} = \varphi(t+\omega)$, for all $t \in \mathbb{R}^+$

so that the function φ is periodic in t with period ω. See also, in Fig. 2.
Moreover,
$$-1 \le \frac{\cos t}{2+\sin 4t} \le 1 \quad \text{for all } t$$
this implies that
$$\varphi(t) \le 2 = M < \infty \quad \text{for all } t.$$

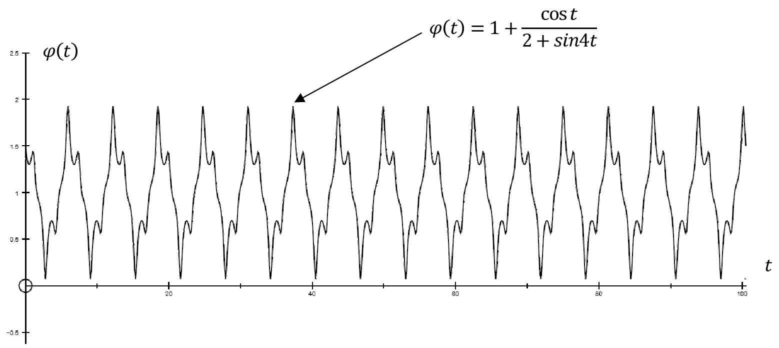

Fig. 2. The periodic function $\varphi(t)$ for all $t \in \mathbb{R}^+$

(vii) Finally, $\psi(t) = 1 + \cos(t/2)$, a periodic function of t (see Fig. 3).

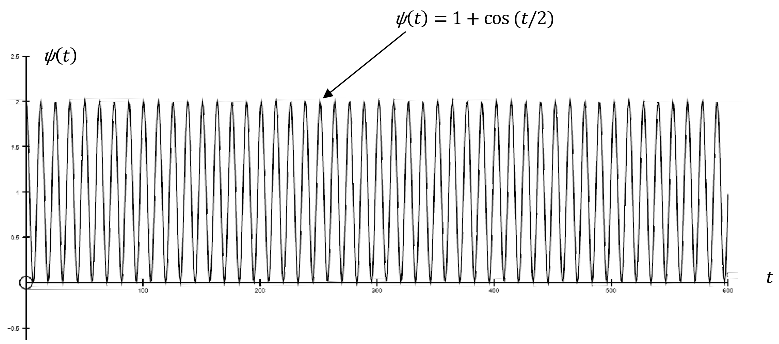

Fig. 3. The periodic function $\psi(t)$ for all $t \in \mathbb{R}^+$

Since $-1 \leq \cos(t/2) \leq 1$ for all t, it follows that $0 \leq \psi(t) \leq 2$ for all t where $2 = \epsilon_1 > 0$

All the assumptions of Theorem 3.1, Theorem 3.2 and Theorem 3.3 are all satisfied, thus by Theorem 3.1 the trivial solution of (3.14) (when $p = 0$) is uniformly asymptotically stable, by Theorem 3.2 the solution $(x(t), y(t), z(t))$ of (3.14) is uniformly ultimately bounded and by Theorems 3.2 and 3.3, system (3.14) has a periodic solution of period ω.

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References

[1] Ademola, A. T., Arawomo, P. O.: *On the asymptotic behaviour of solutions of certain differential equations of the third order.* Proyecciones Journal of Mathematics **33**, 1 (2014), 111–132.

[2] Ademola, A. T., Arawomo, P. O.: *Generalization of some qualitative behaviour of solutions of third order nonlinear differential equations.* Differential Equations and Control Processes **N 1** (2012), 97–113.

[3] Ademola, A. T., Arawomo, P. O.: *Asymptotic behaviour of solutions of third order nonlinear differential equations*. Acta Univ. Sapientiae, Mathematica **3**, 2 (2011), 197–211.

[4] Ademola, A. T., Arawomo, P. O.: *Boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order*. Afr. Mat. **23** (2012), 261–271.

[5] Ademola, A. T., Arawomo, P. O.: *Boundedness and stability of solutions of some nonlinear differential equations of the third-order*. The Pacific Journal of Science and Technology **10**, 2 (2009), 187–193.

[6] Ademola, A. T., Arawomo, P. O.: *Stability and ultimate boundedness of solutions to certain third-order differential equations*. Applied Mathematics E-Notes **10** (2010), 61–69.

[7] Ademola, A. T., Arawomo, P. O.: *Stability and uniform ultimate boundedness of solutions of a third-order differential equation. International Journal of Applied Mathematics* **23**, 1 (2010), 11–22.

[8] Ademola, A. T., Arawomo, P. O.: *Stability and uniform ultimate boundedness of solutions of some third order differential equations*. Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **27** (2011), 51–59.

[9] Ademola, A. T., Arawomo, P. O.: *Stability, boundedness and asymptotic behaviour of solutions of certain nonlinear differential equations of the third order*. Kragujevac J. Math. **35**, 3 (2011), 431–445.

[10] Ademola, A. T., Ogundiran, M. O., Arawomo, P. O., Adesina, O. A.: *Boundedness results for a certain third-order nonlinear differential equations*. Appl. Math. Comput. **216** (2010), 3044–3049.

[11] Afuwape, A. U., Adesina, O. A.: *On the bounds for mean-values of solutions to certain third-order nonlinear differential equations*. Fasciculi Mathematici **36** (2005), 5–14.

[12] Andres, J.: *Boundedness results for solutions of the equation* $\dddot{x}+a\ddot{x}+g(x)\dot{x}+h(x)=p(t)$ *without the hypothesis* $h(x)\operatorname{sgn}x \geq 0$ *for* $|x|>R$. Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **80**, 8 (1986), 533–539.

[13] Antoisewicz, H. A.: *On nonlinear differential equations of the second order with integrable forcing term*. J. London Math. Soc. **30** (1955), 64–67.

[14] Babaev, E. V., Hefferlin, R.: *Concept of chemical periodicity: from Mendeleev Table to Molecular Hyper-periodicity patterns*. In: *Research Studies Press*, London, 1996, 24–81.

[15] Bereketoğlu, H., Györi, I.: *On the boundedness of solutions of a third-order nonlinear differential equation*. Dynam. Systems Appl. **6**, 2 (1997), 263–270.

[16] Callahan, J., Cox, D., Hoffman, K., O'Shea, D., Pollatsek, H., Senechal, L., L.: Calculus in Context. The Five College Calculus Project, *Five Colleges*, 2008.

[17] Chukwu, E. N.: *On boundedness of solutions of third order differential equations*. Ann. Mat. Pura. Appl. **104**, 4 (1975), 123–149.

[18] Ezeilo, J. O. C.: *A note on a boundedness theorem for some third-order differential equations*. J. London Math. Soc. **36** (1961), 439–444.

[19] Ezeilo, J. O. C.: *An elementary proof of a boundedness theorem for a certain third-order differential equation*. J. London Math. Soc. **38** (1963), 11–16.

[20] Ezeilo, J. O. C.: *A boundedness theorem for a certain third-order differential equation*. Proc. London Math. Soc. **13**, 3 (1963), 99–124.

[21] Hara, T.: *On the uniform ultimate boundedness of solutions of certain third-order differential equations*. J. Math. Anal. Appl. **80**, 2 (1981), 533–544.

[22] Mehri, B., Niksirat, M. A.: *On the existence of periodic solutions for certain differential equations*. J. Comp. & Appl. Math. **174** (2005), 239–249.

[23] Mehri, B., Shadman, D.: *Boundedness of solutions of certain third-order differential equation*. Math. Inequal. Appl., 4 (1999), 545–549.

[24] Mehri, B., Shadman, D.: *On the existence of periodic solutions of a certain class of second order nonlinear differential equation.* Math. Inequal. Appl. **1**, 3 (1998), 431–436.

[25] Mehri, B., Shadman, D.: *Periodic solutions of a certain non-linear third order differential equation.* Scientia Iranica **11** (2004), 181–184.

[26] Mehri, B.: *Periodic solution for certain non linear third-order differential equation.* Indian J. Pure Appl. Math. **21**, 3 (1990), 203–210.

[27] Minhós, F.: *Periodic solutions for a third order differential equation under conditions on the potential.* Portugaliae Mathematica **55**, 4 (1998), 475–484.

[28] LaSalle, J., Lefschetz, S.: *Stability by Liapunov's direct method with applications. Academic Press*, New York–London, 1961.

[29] Ogundare, B. S.: *On the boundedness and stability results for the solutions of certain third-order nonlinear differential equations.* Kragujevac J. Math. **29** (2006), 37–48.

[30] Omeike, M. O.: *New result in the ultimate boundedness of solutions of a third-order nonlinear ordinary differential equation.* J. Inequal. Pure and Appl. Math. **9**, 1 (2008), Art. 15, 1–8.

[31] Reissig, R., Sansone, G., Conti, R.: Nonlinear Differential Equations of Higher Order. *Noordhoff International Publishing*, Leyeden, 1974.

[32] Rouche, N., Habets, N., Laloy, M.: Stability Theory by Liapunov's Direct Method. Applied Mathematical Sciences **22**, *Spriger-Verlag*, New York–Heidelberg–Berlin, 1977.

[33] Shadman, D., Mehri, B.: *On the periodic solutions of certain nonlinear third order differential equations.* Z Angew. Math. Mech. **75**, 2 (1995), 164–166.

[34] Swick, K. E.: *On the boundedness and the stability of solutions for some non-autonomous differential equations of the third order.* J. London Math. Soc. **44** (1969), 347–359.

[35] Tejumola, H. O.: *A note on the boundedness of solutions of some nonlinear differential equations of the third-order.* Ghana J. of Science **11**, 2 (1970), 117–118.

[36] Tejumola, H. O.: *A note on the boundedness and the stability of solutions of certain third-order differential equations.* Ann. Math. Pura. Appl. **92** (1972), 65–75.

[37] Tunç, C., Çmar, I.: *On the existence of periodic solutions to nonlinear differential equations of second order.* Differ. Uravn. Protsessy Upr. (Differential equations and control processes), 3 (2008), 1–6.

[38] Tunç, C.: *Boundedness of solutions of a third-order nonlinear differential equation.* J. Inequal. Pure and Appl. Math. **6**, 1 (2005), 1–6.

[39] Tunç, C.: *On existence of periodic solutions to certain nonlinear third order differential equations.* Proyecciones Journal of Mathematics **28**, 2 (2009), 125–132.

[40] Tunç, C.: *Some new results on the boundedness of solutions of a certain nonlinear differential equation of third-order.* International J. of Nonlinear Science **7**, 2 (2009), 246–256.

[41] Yoshizawa, T.: *Liapunov's function and boundedness of solutions.* Funkcialaj Ekvacioj **2** (1958), 71–103.

[42] Yoshizawa, T.: *Stability Theory by Liapunov's Second Method. The Mathematical Society of Japan*, 1966.

[43] Yoshizawa, T.: Stability Theory and Existence of Periodic Solutions and Almost Periodic Solutions. *Spriger-Verlag*, New York–Heidelberg–Berlin, 1975.