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Derivations and Translations on Trellises

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Abstract

G. Szász, J. Szendrei, K. Iseki and J. Nieminen have made an extensive study of derivations and translations on lattices. In this paper, the concepts of meet-translations and derivations have been studied in trellises (also called weakly associative lattices or WA-lattices) and several results in lattices are extended to trellises. The main theorem of this paper, namely, that every derivatrion of a trellis is a meet-translation, is proved without using associativity and it generalizes a well-known result of G. Szász.

Key words: Psoset, trellis, ideal, meet-translation, derivation.

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1 Introduction

Any reflexive and antisymmetric binary relation $\trianglelefteq$ on a set L is called a *pseudo-order* on L and $\langle L; \trianglelefteq\rangle$ is called a *pseudo-ordered set* or a *psoset*. Two elements x and y are comparable if $x \trianglelefteq y$ or $y \trianglelefteq x$. For a subset B of L, the notions of a *lower bound*, an *upper bound*, the *greatest lower bound* (g.l.b. or meet denoted by $\bigwedge B$), the *least upper bound* (l.u.b. or join denoted by $\bigvee B$) are defined analogously to the corresponding notions in a partially ordered set or a poset.

By a *trellis* we mean a psoset, any two of whose elements have a g.l.b. and a l.u.b. Similarly to lattices, trellises can be defined as algebras $\langle L; \vee, \wedge\rangle$ where $\vee, \wedge$ and $\trianglelefteq$ are related as in lattices: a *trellis* is an algebra $\langle L; \vee, \wedge\rangle$ where the binary operations $\vee$ and $\wedge$ satisfy the following properties:

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(i) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$,

(ii) $a \vee (b \wedge a) = a = a \wedge (b \vee a)$,

(iii) $a \vee ((a \wedge b) \vee (a \wedge c)) = a = a \wedge ((a \vee b) \wedge (a \vee c))$.

The notion of a psoset and a trellis are due to E. Fried [1] and H. L. Skala [9]. In [7], it is shown that any psoset can be regarded as a digraph (possibly infinite). A *tournament* is a psoset in which every two elements are comparable. For the undefined notations and terminology, [7] and [9] may be referred.

A *subtrellis* S of a trellis L is a nonempty subset of L such that $a, b \in S$ implies that $a \wedge b$, $a \vee b$ belong to S, where $\wedge$ and $\vee$ are considered in L. An *ideal* I of a trellis L is a subtrellis of L such that $i \in I$ and $a \in L$ imply that $a \wedge i \in I$, or equivalently, $i \in I$, $a \in L$ and $a \trianglelefteq i$ imply that $a \in I$. H. L. Skala in [9] has included the empty set also as an ideal of a trellis. If B is a nonempty subset of a trellis L, then the *ideal generated by* B is defined to be the intersection of all ideals of L containing B and is denoted by $(B]$. The ideal generated by a single element a is called the *principal ideal* generated by a and is denoted by $(a]$. The dual notions are defined similarly. The set of all ideals of a trellis L forms a lattice with respect to set inclusion and it is denoted by $I(L)$. In fact, for $I, J \in I(L)$, $I \wedge J = I \cap J$ and $I \vee J = (I \cup J]$.

2 Meet-translations and derivations on trellises

Definition 2.1 A mapping λ of a trellis L into itself is called a

(i) meet-translation if $\lambda(x \wedge y) = \lambda(x) \wedge y$ for all $x, y \in L$;

(ii) join-translation if $\lambda(x \vee y) = \lambda(x) \vee y$ for all $x, y \in L$.

Examples

(1) The identity mapping of any trellis is both a join-translation and a meet-translation.

(2) If a trellis L with least element 0 has at least two elements, then the mapping w defined by $w(x) = 0$ for every $x \in L$ is a meet-translation that is not a join-translation.

The following lemma and the two propositions generalize the corresponding results in lattices [10] to trellises.

Lemma 2.2 *Let λ be a meet-translation on a trellis L. Then for all $x, y \in L$,*

(i) $x \trianglelefteq y$ implies $\lambda(x) \trianglelefteq \lambda(y)$;

(ii) $\lambda(x) \trianglelefteq x$;

(iii) $\lambda(\lambda(x)) = \lambda(x)$, i.e. λ is idempotent;

(iv) $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$, i.e. λ is a meet-endomorphism;

(v) the fixed elements of λ (x is said to be a fixed element of λ if $\lambda(x) = x$) form an ideal of L which will be called the fixed ideal of λ, denoted by $\operatorname{Fix}\lambda$*; also* $\operatorname{Fix}\lambda = \lambda(L)$.

Proof Follows easily.

Proposition 2.3 *Any two meet-translations of a trellis are permutable (two mappings f and g are said to be permutable if $f \circ g = g \circ f$ where $\circ$ is the composition of mappings).*

Proof Follows easily because $f(g(x)) = f(x \wedge g(x)) = f(x) \wedge g(x)$ for any two meet-translations f, g. □

Remark 2.4 The set of all meet-translations on a trellis L forms a commutative monoid with respect to composition of mappings.

Proposition 2.5 *If λ_1 and λ_2 are any two distinct meet-translations of a trellis L, then* $\operatorname{Fix}\lambda_1 \neq \operatorname{Fix}\lambda_2$.

Proof If $\operatorname{Fix}\lambda_1 = \operatorname{Fix}\lambda_2$, then $\{x \in L \mid \lambda_1(x) = x\} = \{x \in L \mid \lambda_2(x) = x\}$. This implies $\lambda_1(x) = \lambda_1(\lambda_2(x)) = \lambda_2(\lambda_1(x)) = \lambda_2(x)$, a contradiction to the hypothesis that $\lambda_1 \neq \lambda_2$. Therefore $\operatorname{Fix}\lambda_1 \neq \operatorname{Fix}\lambda_2$. □

Proposition 2.6 *If A is an ideal of a trellis L and $\lambda\colon L \to L$ is a meet-translation, then $\lambda(A)$ is an ideal of A and hence an ideal of L.*

Proof By (ii) of Lemma 2.2, $\lambda(A) \subseteq A$. Hence $\lambda{\restriction_A}\colon A \to A$ is also a meet-translation. We easily observe that $\lambda(A) = \{a \in A \mid a = \lambda(a)\}$. This shows that $\lambda(A)$ is the set of all fixed elements of A under the meet-translation $\lambda{\restriction_A}\colon A \to A$. Applying (v) of Lemma 2.2 to $\lambda{\restriction_A}\colon A \to A$ we conclude that $\lambda(A)$ is an ideal of A. Hence an ideal of L. □

Remark 2.7 By (iv) of Lemma 2.2, every meet-translation on a trellis is a meet-endomorphism. G. Szász [10] has proved that every meet-translation of a lattice L is a join-endomorphism if and only if L is distributive.

Remark 2.7 suggests the following open problem.

Problem Characterize those trellises in which every meet-translation is a join-endomorphism.

As every distributive trellis is a lattice [9], it is natural to consider the inequality (2.1) which is valid in tournaments. The following proposition answers the problem partially.

Proposition 2.8 *If a trellis L satisfies the inequality*

$$x \wedge (y \vee z) \trianglelefteq (x \wedge y) \vee (x \wedge z), \tag{2.1}$$

then every meet-translation is a join-endomorphism.

Proof Let L be a trellis satisfying the property (2.1) and λ be a meet-translation on L. For any $x, y \in L$,

$$\begin{aligned}\lambda(x) \vee \lambda(y) &= \lambda((x \vee y) \wedge x) \vee \lambda((x \vee y) \wedge y)\\ &= (\lambda(x \vee y) \wedge x) \vee (\lambda(x \vee y) \wedge y)\\ &\trianglerighteq \lambda(x \vee y) \wedge (x \vee y) && \text{by (2.1)}\\ &= \lambda(x \vee y).\end{aligned}$$

Since $x, y \trianglelefteq x \vee y$, we have $\lambda(x), \lambda(y) \trianglelefteq \lambda(x \vee y)$. Then $\lambda(x) \vee \lambda(y) \trianglelefteq \lambda(x \vee y)$. Therefore $\lambda(x \vee y) = \lambda(x) \vee \lambda(y)$. □

Remark 2.9 The converse of the above proposition is not true. For, the trellis L of Figure 1 has only three meet-translations λ_0, λ_1 and I which are respectively defined by

$$\begin{aligned}\lambda_0(x) &= 0 \text{ for every } x \in L,\\ \lambda_1(x) &= \begin{cases} 0 & \text{for } x \in \{0, a, b\},\\ d & \text{for } x \in \{c, d, 1\},\end{cases}\\ I(x) &= x \text{ for every } x \in L.\end{aligned}$$

Each of these meet-translations is a join-endomorphism, but the trellis does not satisfy (2.1) because $c \wedge (a \vee d) = c \ntrianglelefteq d = (c \wedge a) \vee (c \wedge d)$.

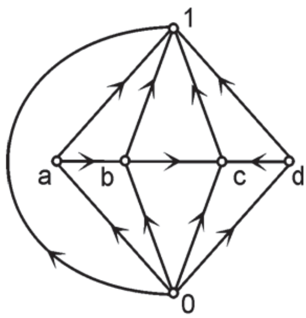

Fig. 1

Definition 2.10 A mapping β of a trellis L into itself is called a derivation of L if it satisfies the following conditions for all $x, y \in L$:

(i) $\beta(x \vee y) = \beta(x) \vee \beta(y)$;

(ii) $\beta(x \wedge y) = (\beta(x) \wedge y) \vee (\beta(y) \wedge x)$.

The mappings given in Examples (1) and (2) are also derivations.

Lemma 2.11 *If β is a derivation on a trellis L, then for all elements $x, y \in L$:*

(i) $x \trianglelefteq y$ implies $\beta(x) \trianglelefteq \beta(y)$;

(ii) $\beta(x) \trianglelefteq x$*;*

(iii) $\beta(\beta(x)) = \beta(x)$*;*

(iv) $x \trianglelefteq y$ *implies* $\beta(x) = x \wedge \beta(y)$*.*

Proof (i) to (iii) follow easily. (iv): Let $x \trianglelefteq y$. Then $\beta(x) \trianglelefteq \beta(y)$ by (i) and $\beta(x) \trianglelefteq x$ by (ii). Therefore $\beta(x) \trianglelefteq x \wedge \beta(y)$. Also

$$\beta(x) = \beta(x \wedge y) = (\beta(x) \wedge y) \vee (\beta(y) \wedge x) \trianglerighteq x \wedge \beta(y).$$

Hence $\beta(x) = x \wedge \beta(y)$. $\square$

Following is the main theorem of this paper generalizing a well-known result that "Every derivatrion of a lattice is a meet-translation" due to G. Szász [10]. The proofs are not similar as $\wedge$ and $\vee$ are not associative in trellises, the theorem is proved without using associativity.

Theorem 2.12 *Every derivation of a trellis L is a meet-translation on L.*

Proof Let β be a derivation of a trellis L. Then by (ii) of Definition 2.10

$$\beta(u \wedge v) \trianglerighteq \beta(u) \wedge v \tag{2.2}$$

for all $u, v \in L$. Taking $x = \beta(u) \wedge v$ and $y = \beta(u)$ in (iv) of Lemma 2.11, we have

$$\begin{aligned} \beta(\beta(u) \wedge v) &= (\beta(u) \wedge v) \wedge \beta(\beta(u)) \\ &= (\beta(u) \wedge v) \wedge \beta(u) \qquad \text{by (iii) of Lemma 2.11} \\ &= \beta(u) \wedge v. \end{aligned}$$

Thus

$$\beta(\beta(u) \wedge v) = \beta(u) \wedge v \tag{2.3}$$

which gives us $(\beta(u) \wedge v) \vee (\beta(u) \wedge \beta(v)) = \beta(u) \wedge v$ implying

$$\beta(u) \wedge v \trianglerighteq \beta(u) \wedge \beta(v). \tag{2.4}$$

Since $\beta(u) \wedge v \trianglelefteq v$, by (i) of Lemma 2.11, $\beta(\beta(u) \wedge v) \trianglelefteq \beta(v)$. Then, by (2.3), $\beta(u) \wedge v \trianglelefteq \beta(v)$. Also $\beta(u) \wedge v \trianglelefteq \beta(u)$. Therefore

$$\beta(u) \wedge v \trianglelefteq \beta(u) \wedge \beta(v). \tag{2.5}$$

From (2.4) and (2.5), $\beta(u) \wedge v = \beta(u) \wedge \beta(v)$. Thus

$$\beta(u) \wedge v = \beta(u) \wedge \beta(v) \trianglerighteq \beta(u \wedge v) \tag{2.6}$$

since $u \wedge v \trianglelefteq u, v$ implies $\beta(u \wedge v) \trianglelefteq \beta(u), \beta(v)$ which in turn implies $\beta(u \wedge v) \trianglelefteq \beta(u) \wedge \beta(v)$. From (2.2) and (2.6), $\beta(u \wedge v) = \beta(u) \wedge v$ for all $u, v \in L$, so that β is a meet-translation. $\square$

By G. Szász [10], Corollary 3, every derivation on a lattice L is of the form $\beta(x) = x \wedge c$ for some $c \in L$ if and only if L has greatest element. This corollary holds in trellises by Lemma 2.11 (iv).

Remark 2.13 The converse of Theorem 2.12 is not true. For, in the lattice of Figure 2, the mapping $\lambda\colon L \to L$ defined by $\lambda(0) = 0 = \lambda(z)$, $\lambda(x) = x$, $\lambda(y) = y$ and $\lambda(1) = y$ is a meet-translation. It is not a join-endomorphism because $\lambda(x \vee z) \neq \lambda(x) \vee \lambda(z)$.

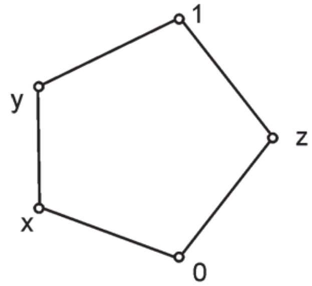

Fig. 2

The following theorem can be easily proved.

Theorem 2.14 *A meet-translation λ of a trellis L is a derivation on L if and only if λ is a join-endomorphism.*

Remark 2.15 Every meet-translation of a trellis L satisfying the inequality (2.1) is a derivation on L.

Remark 2.16 The set of all derivations on a trellis L forms a commutative monoid with respect to composition of mappings.

3 On the set of all meet-translations on a trellis

G. Szász and J. Szendrei [11] have proved that the set of all meet-translations on a lattice L forms a meet-semilattice. The next theorem generalizes this result to a trellis L.

Let $\Phi(L)$ be the set of all meet-translations on a trellis L. The binary relation $\leq$ on $\Phi(L)$ defined by, for $\lambda_1, \lambda_2 \in \Phi(L)$, $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1(x) \trianglelefteq \lambda_2(x)$ for every $x \in L$, is a partial order on $\Phi(L)$. Reflexivity and antisymmetry of $\leq$ follow easily. If $\lambda_1, \lambda_2, \lambda_3 \in \Phi(L)$ are such that $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1(x) = \lambda_1(x) \wedge \lambda_2(x) = \lambda_1(\lambda_2(x))$ and $\lambda_2(x) = \lambda_2(x) \wedge \lambda_3(x)$, whence $\lambda_1(x) \wedge \lambda_3(x) = \lambda_1(\lambda_2(x)) \wedge \lambda_3(x) = \lambda_1(\lambda_2(x) \wedge \lambda_3(x)) = \lambda_1(\lambda_2(x)) = \lambda_1(x)$ for any $x \in L$, thus $\lambda_1 \leq \lambda_3$.

The identity mapping I is the greatest element of the poset $\langle \Phi(L); \leq \rangle$. If the trellis L has the least element 0, then the mapping $\lambda_0\colon L \to L$ defined by $\lambda_0(x) = 0$ for every $x \in L$ is the least element of $\langle \Phi(L); \leq \rangle$.

Let L be a trellis and $f\colon \Phi(L) \to I(L)$ be the mapping defined by $f(\lambda) = \operatorname{Fix} \lambda$ for $\lambda \in \Phi(L)$. Then f is one-to-one by Proposition 2.5. However f need not be onto. For, in the trellis of Figure 3, there are only two meet-translations, namely, the identity mapping I and the mapping λ_0 defined by $\lambda_0(x) = 0$ for every $x \in L$. Now, $\mathrm{Fix}\, I = L$ and $\mathrm{Fix}\, \lambda_0 = \{0\}$. Therefore, for the ideals $\{0, a\}$ and $\{0, a, b\}$ belonging to $I(L)$ there are no pre-images in $\Phi(L)$.

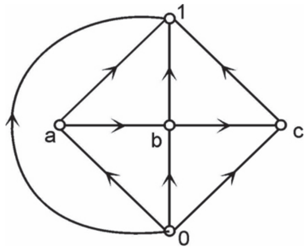

Fig. 3

The fact that f is isotone is trivial, as $\lambda_1 \leq \lambda_2$ obviously implies $\mathrm{Fix}\, \lambda_1 \subseteq \mathrm{Fix}\, \lambda_2$.

Theorem 3.1 *Let $\Phi(L)$ be the set of all meet-translations on a trellis L. Then $\langle \Phi(L); \leq \rangle$ is a meet-semilattice.*

Proof In the poset $\langle \Phi(L); \leq \rangle$, clearly $\lambda_1 \circ \lambda_2 \in \Phi(L)$ whenever $\lambda_1, \lambda_2 \in \Phi(L)$. Also $\lambda_1 \circ \lambda_2$ is the g.l.b. of λ_1, λ_2 since $(\lambda_1 \circ \lambda_2)(x) = \lambda_1(x) \wedge \lambda_2(x)$. Thus $\langle \Phi(L); \leq \rangle$ is a meet-semilattice. □

It is known that if L is a distributive lattice, then $\Phi(L)$ forms a lattice [6]. The following problem naturally arises and remains open:

Problem Characterize trellises L for which $\Phi(L)$ forms a lattice.

Proposition 3.2 *If a trellis L is a cycle, then L has exactly one meet-translation (derivation) and it is the identity mapping.*

Proof Let the trellis L be a cycle. Let $\Phi(L)$ be the set of all meet-translations on L. Define a mapping $f\colon \Phi(L) \to I(L)$ by $f(\lambda) = \mathrm{Fix}\, \lambda$ for every $\lambda \in \Phi(L)$. f is a one-to-one mapping by Proposition 2.5. The identity mapping is a meet-translation of L. If $\lambda_1 \neq I$ is any meet-translation, then $\mathrm{Fix}\, \lambda_1 \neq \mathrm{Fix}\, I = L$ in $I(L)$, which is not possible as L is the only ideal of L. Thus the identity mapping is the only meet-translation. □

Remark 3.3 The converse of the above proposition is not true. For, in the trellis of Figure 4, the identity mapping is the only meet-translation, but the trellis is not a cycle.

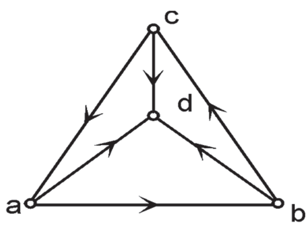

Fig. 4

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