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Cyclic Type Fixed Point Results in 2-Menger Spaces

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Abstract

In this paper we introduce generalized cyclic contractions through r number of subsets of a probabilistic 2-metric space and establish two fixed point results for such contractions. In our first theorem we use the Hadzic type t-norm. In another theorem we use a control function with minimum t-norm. Our results generalizes some existing fixed point theorem in 2-Menger spaces. The results are supported with some examples.

Key words: 2-Menger space, Cauchy sequence, fixed point, control function, t-norm.

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1 Introduction and mathematical preliminaries

In 1922, S. Banach [1] established the well known Banach contraction principle. This celebrated work has been generalized by many authors in various spaces [6, 8, 11, 26]. In particular, the various fixed point theorems are used to demonstrate the existence and uniqueness of a solution of differential equation, integral equation, functional equation, partial differential equation and others.

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The following definitions are used in our main results.

The concept of metric spaces has been extended in various ways. One such extension has been made by Gähler [15] in which a positive real number is assigned to every three elements of the space.

Definition 1.1 (2-metric space [15, 16]) Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(i) given distinct elements x, y of X, there exists an element z of X such that $d(x, y, z) \neq 0$,

(ii) $d(x, y, z) = 0$ when at least two of x, y, z are equal,

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X and

(iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X. When d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.

The following is the example of 2-Metric space.

Example 1.1 [37] Let R^2 be the Euclidean space. Let $d(x, y, z)$ denote the area of the triangle formed by joining the three points $x, y, z \in R^2$. Then (R^2, d) is a 2-metric space.

Fixed point theory has developed rapidly in these spaces. Several results of metric fixed point theory was extended to these spaces. Some of the important fixed point theorems in 2-metric spaces are [21, 22, 26, 28, 29, 30, 32, 37].

Definition 1.2 [20, 35] A mapping $F \colon R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with

$$\inf_{t \in R} F(t) = 0 \quad \text{and} \quad \sup_{t \in R} F(t) = 1,$$

where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

An interpretation of $F_{x,y}(t)$ is that it is the probability of the event that the distance between the points x and y is less than t. A metric space becomes a Menger space if we write $F_{x,y}(t) = H(t - d(x, y))$ where H is the Heaviside function given by

$$H(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Probabilistic metric spaces are probabilistic generalizations of metric spaces in which every pair of elements is assigned to a distribution function. The theory of these spaces is an important part of stochastic analysis. Schweizer and Sklar in their book noted in [35] have given a comprehensive account of several aspects of such spaces.

Definition 1.3 (Probabilistic metric space [20, 35]) A probabilistic metric space (briefly, a PM-space) is an ordered pair (X, F), where X is a non empty set and F is a mapping from $X \times X$ into the set of all distribution functions. We denote the distribution function $F(x, y)$ by $F_{x,y}$. $F_{x,y}(t)$ represents the value of $F_{x,y}$ at $t \in R$. The function $F_{x,y}$ is assumed to satisfy the following conditions for all $x, y \in X$:

(i) $F_{x,y}(0) = 0$,

(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$,

(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(t_1 + t_2) = 1$.

A particular type of probabilistic metric space is Menger space in which the triangular inequality is postulated with the help of a t-norm.

Shi, Ren and Wang introduced the concept of n-th order t-norm in 2003.

Definition 1.4 (n-th order t-norm [39]) A mapping $T\colon \Pi_{i=1}^{n}[0,1] \to [0,1]$ is called a n-th order t-norm if the following conditions are satisfied:

(i) $T(0, 0, \ldots, 0) = 0$, $T(a, 1, 1, \ldots, 1) = a$ for all $a \in [0, 1]$,

(ii) $T(a_1, a_{2,}, a_3, \ldots, a_n) = T(a_2, a_1, a_3, \ldots, a_n) = T(a_2, a_3, a_1, \ldots, a_n)$
$= \ldots = T(a_2, a_3, a_4, \ldots, a_n, a_1)$,

(iii) $a_i \geq b_i$, $i = 1, 2, 3, \ldots, n$ implies $T(a_1, a_2, a_3, \ldots, a_n) \geq T(b_1, b_2, b_3, \ldots, b_n)$,

(iv) $T(T(a_1, a_2, a_3, \ldots, a_n), b_2, b_3, \ldots, b_n)$
$= T(a_1, T(a_2, a_3, \ldots, a_n, b_2), b_3, \ldots, b_n)$
$= T(a_1, a_2, T(a_3, a_4, \ldots, a_n, b_2, b_3), b_4, \ldots, b_n)$
$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$= T(a_1, a_2, \ldots, a_{n-1}, T(a_n, b_2, b_3, \ldots, b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

Definition 1.5 (t-norm [20, 35]) A t-norm is a function $\Delta\colon [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions for all $a, b, c, d \in [0, 1]$

(i) $\Delta(1, a) = a$,

(ii) $\Delta(a, b) = \Delta(b, a)$,

(iii) $\Delta(c, d) \geq \Delta(a, b)$ whenever $c \geq a$ and $d \geq b$,

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

The following are three examples of 3rd order t-norm:

(i) The minimum t-norm, $\Delta = T_m$, defined by $T_m(a, b, c) = \min\{a, b, c\}$.

(ii) The product t-norm, $\Delta = T_p$, defined by $T_p(a, b, c) = a \cdot b \cdot c$.

(iii) The Lukasiewicz t-norm, $\Delta = T_L$, defined by

$$T_L(a, b, c) = \max\{a + b + c - 2, 0\}.$$

Definition 1.6 (Hadzic type t-norm [20]) A t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0,1)$ as

$$\Delta^0(s) = 1,\ \Delta^{p+1}(s) = \Delta(\Delta^p(s), s) \quad \text{for all integer } p \geq 0,$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exits $\eta(\lambda) \in (0,1)$ such that $1 \geq s > \eta(\lambda) \Rightarrow \Delta^p(s) \geq 1 - \lambda$ for all integer $p \geq 0$.

Here we use the Hadzic type t-norm and 3rd-order minimum t-norm.

Definition 1.7 (Menger space [20, 35]) A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in X$ if and only if $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X$, $s > 0$ and

(iv) $F_{x,y}(u + v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ for all $u, v \geq 0$ and $x, y, z \in X$.

The first fixed point result in probabilistic metric spaces proved by Sehgal and Bharucha-Reid [36]. After that a lot of results appeared in the literature. A comprehensive survey upto 2001 is given by Hadzic and Pap in [20].

Probabilistic generalization of 2-metric spaces has been done following the same ideas behind the introduction of probabilistic metric spaces.

Definition 1.8 (Probabilistic 2-metric space [42]) A probabilistic 2-metric space is an order pair (X, F) where X is an arbitrary set and F is a mapping from X^3 into the set of distribution functions. The distribution function $F_{x,y,z}(t)$ will denote the value of $F_{x,y,z}$ at the real number t. The function $F_{x,y,z}$ are assumed to satisfy the following conditions:

(i) $F_{x,y,z}(t) = 0$ for all $t \leq 0$ and for all $x, y, z \in X$,

(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of the three points x, y, z are equal,

(iii) for distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ if $t > 0$,

(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$,

(v) $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then

$$F_{x,y,z}(t_1 + t_2 + t_3) = 1.$$

Example 1.2 $F_{x,y,z}(t) = \begin{cases} \frac{t}{t+\min\{|x-y|,|x-z|,|y-z|\}}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$

A special case of the above definition is the following.

Definition 1.9 (2-Menger space [38]) Let X be any nonempty set and D the set of all left-continuous distribution functions. A triplet (X, F, Δ) is said to be a 2-Menger space if F is a mapping from X^3 into D satisfying the following conditions where the value of F at $x, y, z \in X^3$ is represented by $F_{x,y,z}$ or $F(x, y, z)$ for all $x, y, z \in X$ such that

(i) $F_{x,y,z}(0) = 0$,

(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of $x, y, z \in X$ are equal,

(iii) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$, for all $x, y, z \in X$,

(iv) $F_{x,y,z}(t) \geq \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$

where $t_1 + t_2 + t_3 = t$ and $x, y, z, w \in X$ where Δ is the 3rd order t-norm.

Recently many authors established many fixed point results in 2-Menger spaces. The references [5, 8, 18, 19, 40] are some fixed point results on those spaces.

Definition 1.10 [19] A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be converge with limit x if $\lim_{n\to\infty} F_{x_n,x,a}(t) = 1$ for all $t > 0$ and for every $a \in X$.

Definition 1.11 [19] A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be a Cauchy sequence in X if given $\epsilon > 0$, $\lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x_m,a}(\epsilon) \geq 1 - \lambda \tag{1.1}$$

for all $m, n > N_{\epsilon,\lambda}$ and for every $a \in X$.

Definition 1.11 can be equivalently written by replacing '$\geq$' with '$>$' in (1.1). More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definition 1.12 [19] A 2-Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.

In recent years cyclic contraction and cyclic contractive type mapping have appeared in several works.

Definition 1.13 Let A and B be two non-empty sets. A cyclic mapping is a mapping $T \colon A \cup B \to A \cup B$ which satisfies:

$$TA \subseteq B \quad \text{and} \quad TB \subseteq A.$$

This line of research was initiated by Kirk, Srinivasan and Veeramani [25], where they, amongst other results, established the following generalization of the contraction mapping principle.

Theorem 1.1 [25] *Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f \colon A \cup B \to A \cup B$ satisfies:*

(1) $fA \subseteq B$ *and* $fB \subseteq A$*,*

(2) $d(fx, fy) \leq kd(x, y)$ *for all* $x \in A$ *and* $y \in B$ *where* $k \in (0, 1)$*.*

Then f *has a unique fixed point in* $A \cap B$*.*

The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems in probabilistic metric and 2-probabilistic metric spaces may be noted in [6, 7, 10, 11, 17, 23, 41, 43] and [44]. A cyclic contraction result in generalized menger space was established by the recent result of Choudhury, Das and Bhandari [12].

A generalization of cyclic mapping is p-cyclic mapping. The definition is the following:

Definition 1.14 Let $\{A_i\}_{i=1}^{p}$ be non-empty sets. A p-cyclic mapping is a mapping $T\colon \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ which satisfies the following conditions:

(i) $TA_i \subseteq A_{i+1}$ for $1 \leq i < p$, $TA_p \subseteq A_1$.

In this case where $p = 2$, this reduces to cyclic mappings. Some fixed point results of p-cyclic maps have been obtained in [13, 41].

In [24] Khan, Swaleh and Sessa introduced a new category of contractive fixed point problems in metric space. They introduced the concept of "altering distance function", which is a control function that alters the distance between two points in a metric space. This concept was further generalized in a number of works. There are several works in fixed point theory involving altering distance function, some of these are noted in [31, 33] and [34].

Recently Choudhury and Das had extended the concept of altering distance function in the context of Menger spaces in [2]. The definition is as follows:

Definition 1.15 (Φ-function [2]) A function $\phi\colon R \to R^{+}$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,

(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,

(iii) ϕ is left continuous in $(0, \infty)$,

(iv) ϕ is continuous at 0.

In [2] Choudhury and Das introduced a new type of contraction mapping in Menger spaces which is known as ϕ-contraction. Recently Choudhury, Das and Bhandari introduce ϕ-contraction in the context of 2-Menger spaces.

Definition 1.16 [8] Let (X, F, Δ) be a 2-Menger space. A self map $f\colon X \to X$ is said to be ϕ-contractive if

$$F_{fx,fy,a}(\phi(t)) \geq F_{x,y,a}\left(\phi\left(\tfrac{t}{c}\right)\right) \tag{1.2}$$

where $0 < c < 1$, $, x, y \in X$ and $t > 0$, for all $a \in X$ and the function ϕ is a Φ-function.

The idea of control function has opened possibilities of proving new fixed point results in Menger spaces. This concept has also applied to a coincidence point result. Some recent results using Φ-function are noted in [3, 4, 5, 6, 7, 8, 9, 10, 11, 14] and [27].

The purpose of this paper is to establish special type fixed point results in 2-Menger space in which two different types t-norm is used. A control function is also used in one of the theorems.

Definition 1.17 Let (X, F, Δ) be a complete 2-Menger space, where Δ is the 3rd order t-norm. Let $\{A_i\}_{i=1}^r$ be non-empty closed subsets of X such that the mapping $T\colon \bigcup_{i=1}^r A_i \to \bigcup_{i=1}^r A_i$ satisfies the following conditions:

$$TA_i \subseteq A_{i+1} \quad \text{for } 1 \le i < r,\ TA_r \subseteq A_1 \tag{1.3}$$

then the mapping T is called an r-cyclic mapping in 2-Menger space.

The following are illustrations of the above definitions.

Example 1.3 $X = R$,

$$F_{x,y,z}(t) = \begin{cases} \frac{t}{t+\min\{|x-y|,|x-z|,|y-z|\}}, & \text{if } t > 0, \\ 0, & \text{if } t \le 0, \end{cases}$$

$A_1 = [-2, 0] = A_3$ and $A_2 = [0, 2] = A_4$ and $Y = \bigcup_{i=1}^4 A_i$. Define $T\colon Y \to Y$ by $Tx = \frac{-x}{4}$ for all $x \in Y$. It is easily verified that $T(A_i) \subset A_{i+1}$ for $i = 1, 2, 3$ and $T(A_4) \subset A_1$, so that T is an r-cyclic mapping.

Example 1.4 Let $X = \{\frac{1}{n}\} \bigcup \{0\}$ with F defined as in Example 1.3. Now we consider the following subsets of X:

$$A_1 = \left\{\frac{1}{n} \setminus n \text{ is odd}\right\} \cup \{0\} \qquad \text{and} \qquad A_2 = \left\{\frac{1}{n} \setminus n \text{ is even}\right\} \cup \{0\}.$$

Consider the mapping $T\colon X \to X$ given by

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n},\ n \in N. \end{cases}$$

Now A_1 and A_2 are closed and $X = \bigcup_{i=1}^2 A_i$ is a cyclic representation of X with respect to T.

It may be easily examined that the mapping T is an r-cyclic mapping where $r = 2$.

2 Main results

Theorem 2.1 *Let (X, F, Δ) be a complete 2-Menger space with a Hadzic type t-norm Δ such that whenever $x_n \to x$ and $y_n \to y$, for all $a \in X$ and*

$$F_{x_n,y_n,a}(t) \to F_{x,y,a}(t).$$

Let T be an r-cyclic mapping (Definition 1.17) with Hadiz type t-norm in 2-Menger space which satisfies the following conditions:

$$F_{T_x,T_y,a}(t) \geq F_{x,y,a}\left(\frac{t}{k}\right) \tag{2.1}$$

whenever $x \in A_i$, $y \in A_{i+1}$, for all $a \in X$, $k \in (0,1)$, $t > 0$. Then $\bigcap_{i=1}^{r} A_i$ is non-empty and T has a unique fixed point in $\bigcap_{i=1}^{r} A_i$.

Proof Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}$, $n \in N$ where N is the set of natural numbers.

By (1.3), we have

$$\begin{aligned} &x_o \in A_1,\ x_1 \in A_2,\ x_2 \in A_3,\ \ldots,\ x_{r-1} \in A_r \text{ and in general} \\ &x_{nr} \in A_1,\ x_{nr+1} \in A_2,\ \ldots,\ x_{nr+(r-1)} \in A_r\ n \geq 0. \end{aligned} \tag{2.2}$$

For any $n \geq 1$ and for all $a \in X$, $t > 0$, we have

$$F_{x_n,x_{n+1},a}(t) = F_{Tx_{n-1},Tx_n,a}(t) \geq F_{x_{n-1},x_n,a}\left(\frac{t}{k}\right)\ (x_{n-1} \in A_n,\ x_n \in A_{n+1}). \tag{2.3}$$

By successive application of the above inequality, for all $a \in X$, we have for $t > 0$ and $n \geq 0$,

$$F_{x_n,x_{n+1},a}(t) \geq F_{x_0,x_1,a}\left(\frac{t}{k^n}\right). \tag{2.4}$$

Taking $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(t) = 1. \tag{2.5}$$

Again, by repeated applications of (2.3), it follows that for all $a \in X$, $t > 0$ and $n \geq 0$ and each $i \geq 1$,

$$F_{x_{n+i},x_{n+i+1},a}(t) \geq F_{x_n,x_{n+1},a}\left(\frac{t}{k^i}\right). \tag{2.6}$$

We next prove that $\{x_n\}$ is a Cauchy sequence (Definition 1.11), that is, we prove that for arbitrary $\epsilon > 0$ and $0 < \lambda < 1$, there exists $N(\epsilon, \lambda)$ such that for all $a \in X$,

$$F_{x_n,x_m,a}(\epsilon) \geq 1 - \lambda \quad \text{for all } m, n \geq N(\epsilon, \lambda).$$

Without loss of generality we can assume that $m > n$.

Now,

$$\epsilon = \epsilon\frac{1-k}{1-k} > \epsilon(1-k)(1 + k + k^2 + \cdots + k^{m-n-1}).$$

Then, by the monotone increasing property of F, and for all $a \in X$, we have

$$F_{x_n,x_m,a}(\epsilon) \geq F_{x_n,x_m,a}(\epsilon(1-k)(1 + k + k^2 + \cdots + k^{m-n-1})),$$

that is,

$$F_{x_n,x_m,a}(\epsilon) \geq \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), \Delta(F_{x_{n+1},x_{n+2},a}(\epsilon k(1-k)), \Delta(\ldots, \Delta(F_{x_{m-2},x_{m-1},a}(\epsilon k^{m-n-2}(1-k)), F_{x_{m-1},x_m,a}(\epsilon k^{m-n-1}(1-k)))\ldots))). \tag{2.7}$$

Putting $t = (1-k)\epsilon k^i$ in (2.6), for all $a \in X$, we get

$$F_{x_{n+i},x_{n+i+1},a}((1-k)\epsilon k^i) \geq F_{x_n,x_{n+1},a}((1-k)\epsilon).$$

Then, by (2.7), for all $a \in X$, we have

$$F_{x_n,x_m,a}(\epsilon) \geq \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), \Delta(\ldots, \Delta(F_{x_n,x_{n+1},a}(\epsilon(1-k)), F_{x_n,x_{n+1},a}(\epsilon(1-k)))\ldots))),$$

that is,

$$F_{x_n,x_m,a}(\epsilon) \geq \Delta^{(m-n)} F_{x_n,x_{n+1},a}(\epsilon(1-k)). \tag{2.8}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equi-continuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) \geq 1 - \lambda \quad \text{whenever } \eta(\lambda) < s \leq 1. \tag{2.9}$$

Since, $F_{x_0,x_1,a}(t) \to 1$ as $t \to \infty$ and $0 < k < 1$, there exists an positive integer $N(\epsilon, \lambda)$ such that for all $a \in X$,

$$F_{x_0,x_1,a}\left(\frac{(1-k)\epsilon}{k^n}\right) > \eta(\lambda) \quad \text{for all } n \geq N(\epsilon,\lambda). \tag{2.10}$$

From (2.10) and (2.6), with $n = 0$, $i = n$ and $t = (1-k)\epsilon$, for all $a \in X$, we get

$$F_{x_n,x_{n+1},a}(\epsilon(1-k)) > F_{x_0,x_1,a}\left(\frac{(1-k)\epsilon}{k^n}\right) > \eta(\lambda) \quad \text{for all } n \geq N(\epsilon,\lambda).$$

Then, from (2.9) with $s = F_{x_n,x_{n+1},a}(\epsilon(1-k))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1},a}(\epsilon(1-k))) \geq 1 - \lambda.$$

It then follows from (2.8) that for all $a \in X$,

$$F_{x_n,x_m,a}(\epsilon) \geq 1 - \lambda \quad \text{for all } m, n \geq N(\epsilon,\lambda).$$

Thus $\{x_n\}$ is a Cauchy sequence.

By the construction of the sequence $\{x_n\}$, we have $x_r \in A_1$, $x_{2r} \in A_1$, $\ldots$, $x_{nr} \in A_1$. Therefore the subsequence $\{x_{nr}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{nr+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, $\ldots$, A_r are closed sets, similarly we get $z \in A_3$, A_4, $\ldots$, A_r. Therefore $z \in A_1 \cap A_2 \cap A_3 \cdots \cap A_r$.

Now, we prove that $Tz = z$. For this we have

$$F_{z,Tz,a}(t) \geq \Delta(F_{z,Tz,x_n}(s_1), F_{z,x_n,a}(s_2), F_{x_n,Tz,a}(t - s_1 - s_2)), \tag{2.11}$$

(where $s_1, s_2 > 0$ and $t > s_1 + s_2$).

Now, we can choose $\xi > 0$ such that $t - s_1 - s_2 = \xi$. Using the above assumption, from (2.11), we get

$$\begin{aligned} F_{z,Tz,a}(t) &\geq \Delta(F_{z,Tz,Tx_{n-1}}(s_1), F_{z,x_n,a}(s_2), F_{Tx_{n-1},Tz,a}(\xi)) \\ &= \Delta(F_{Tz,Tx_{n-1},z}(s_1), F_{z,x_n,a}(s_2), F_{Tz,Tx_{n-1},a}(\xi)). \end{aligned}$$

Using (1.3) and (2.1), we get

$$F_{z,Tz,a}(t) \geq \Delta\left(F_{z,x_{n-1},z}\left(\frac{s_1}{k}\right), F_{z,x_n,a}(s_2), F_{z,x_{n-1},a}\left(\frac{\xi}{k}\right)\right). \tag{2.12}$$

[since $z \in A_{n-1}, x_{n-1} \in A_n$]

Taking limit as $n \to \infty$ in (2.12), by virtue of the properties of F, we get $F_{z,Tz,a}(t) \geq \Delta(1,1,1) = 1$. Therefore, $F_{z,Tz,a}(t) = 1$.

Hence $z = Tz$, that is, z is a fixed point of T in $A_1 \cap A_2 \cap A_3 \cdots \cap A_r$.

Let v be another fixed point of T, that is, $Tv = v$. Now,

$$F_{z,v,a}(t) = F_{Tz,Tv,a}(t) \geq F_{z,v,a}\left(\frac{t}{k}\right) = F_{Tz,Tv,a}\left(\frac{t}{k}\right) \geq F_{z,v,a}\left(\frac{t}{k^2}\right).$$

Repeating this process n times, we get

$$F_{z,v,a}(t) = F_{Tz,Tv,a}(t) \geq F_{z,v,a}\left(\frac{t}{k^n}\right).$$

Letting $n \to \infty$ on both sides we get from the above inequality,

$$F_{z,v,a}(t) \geq F_{z,v,a}\left(\frac{t}{k^n}\right) \to 1.$$

Hence, $F_{z,v,a}(t) = 1$, which implies that $z = v$.

Hence T have a unique fixed point in $A_1 \cap A_2 \cap A_3 \cdots \cap A_r$. □

Taking $r = 2$ we get the following corollary.

Corollary 2.1 *Let (X, F, Δ) be a complete 2-Menger space with the Hadzic type t-norm Δ and let there exist two non-empty closed subsets A and B of X such that the mapping $T\colon A \cup B \to A \cup B$ which satisfies the following conditions:*

(i) $TA \subseteq B$ and $TB \subseteq A$

and

(ii) $F_{Tx,Ty,a}(t) \geq F_{x,y,a}(\frac{t}{k})$

for all $x \in A$, $y \in B$ and a is an element, different from x, y where $0 < k < 1$. Then $A \cap B$ is non-empty and T has a unique fixed point in $A \cap B$.

Example 2.1 Let $X = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \beta, \gamma\}$, $B = \{\gamma, \delta\}$, the t-norm Δ is a 3rd order minimum t-norm and F be defined as

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.40, & \text{if } 0 < t < 7, \\ 1, & \text{if } t \geq 7, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.95, & \text{if } 0 < t < 1, \\ 1, & \text{if } t \geq 1, \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $T\colon A \cup B \to A \cup B$ as follows: $T\alpha = \gamma$, $T\beta = \delta$, $T\gamma = \gamma$, $T\delta = \gamma$ then the mappings T satisfies all the conditions of the Corollary 2.1, for $k = 0.5$, where γ is the unique fixed point of T in $A \cap B$.

We use the control function ϕ (Definition 1.15) in our next theorem in the inequality (2.1). Here we use the 3rd-order minimum t-norm. We also prove our second theorem by different arguments from the first theorem.

Theorem 2.2 *Let T be an r-cyclic mapping (Definition 1.17) in 2-Menger space with 3rd-order minimum t-norm which satisfies the following conditions:*

$$F_{T_x,T_y,a}(\phi(t)) \geq F_{x,y,a}\left(\phi\left(\frac{t}{c}\right)\right) \tag{2.13}$$

whenever $x \in A_i$, $y \in A_{i+1}$, for all $a \in X$, $c \in (0,1)$, $t > 0$ and ϕ is a ϕ-function (Definition 1.15). Then $\bigcap_{i=1}^{r} A_i$ is non-empty and T has a unique fixed point in $\bigcap_{i=1}^{r} A_i$.

Proof Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}$, $n \in N$ where N is the set of natural numbers.

By (1.3), we have $x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$, $\ldots$, $x_{r-1} \in A_r$ and in general

$$x_{nr} \in A_1,\ x_{nr+1} \in A_2,\ \ldots,\ x_{nr+(r-1)} \in A_r \text{ for all } n \geq 0. \tag{2.14}$$

For any $n \geq 1$ and for all $a \in X$, $t > 0$, we have

$$\begin{aligned} F_{x_n,x_{n+1},a}(\phi(t)) &= F_{Tx_{n-1},Tx_n,a}(\phi(t)) \\ &\geq F_{x_{n-1},x_n,a}\left(\phi\left(\frac{t}{c}\right)\right)\ (x_{n-1} \in A_n,\ x_n \in A_{n+1}). \end{aligned} \tag{2.15}$$

By successive application of the above inequality, we have for $t > 0$ and $n \geq 0$,

$$F_{x_n,x_{n+1},a}(\phi(t)) \geq F_{x_0,x_1,a}\left(\phi\left(\frac{t}{c^n}\right)\right).$$

Taking $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(\phi(t)) = 1. \tag{2.16}$$

Again, by virtue of a property of ϕ and F given $s > 0$, we can find $t > 0$ such that $s > \phi(t)$. Thus the above limit implies that for all $s > 0$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(s) = 1. \tag{2.17}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\epsilon) < 1 - \lambda. \tag{2.18}$$

We take $m(k)$ corresponding to $n(k)$ to be the smallest integer satisfying (2.18), so that

$$F_{x_{m(k)-1},x_{n(k)},a}(\epsilon) \geq 1 - \lambda. \tag{2.19}$$

If $\epsilon_1 < \epsilon$ then we have

$$F_{x_{m(k)},x_{n(k)},a}(\epsilon_1) \leq F_{x_{m(k)},x_{n(k)},a}(\epsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $m(k) > n(k) > k$ and satisfying (2.18), (2.19) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\phi(\epsilon_2)) < 1 - \lambda, \tag{2.20}$$

and

$$F_{x_{m(k)-1},x_{n(k)},a}(\phi(\epsilon_2)) \geq 1 - \lambda. \tag{2.21}$$

Now, from (2.20), we get

$$1 - \lambda > F_{x_{m(k)},x_{n(k)},a}(\phi(\epsilon_2))$$

$$\geq \Delta(F_{x_{m(k)},x_{n(k)},x_{m(k)-1}}(\epsilon^{'}), F_{x_{m(k)},x_{m(k)-1},a}(\epsilon^{''}), F_{x_{m(k)-1},x_{n(k)},a}(\phi(\epsilon_2) - \epsilon^{'} - \epsilon^{''}))$$

(where $\epsilon^{'}, \epsilon^{''} > 0$ and $\epsilon^{'} + \epsilon^{''} < \phi(\epsilon_2)$)

$$= \Delta(F_{x_{m(k)},x_{m(k)-1},x_{n(k)}}(\epsilon^{'}), F_{x_{m(k)},x_{m(k)-1},a}(\epsilon^{''}), F_{x_{m(k)-1},x_{n(k)},a}(\phi(\epsilon_2) - \epsilon^{'} - \epsilon^{''})). \tag{2.22}$$

Now by (2.17) for sufficiently large k and by the property of ϕ, we can get $0 < \epsilon^{'}, \epsilon^{''} < \phi(\epsilon_2)$ and $0 < \lambda_1, \lambda_2 < \lambda$ such that

$$F_{x_{m(k)},x_{m(k)-1},x_{n(k)}}(\epsilon^{'}) \geq 1 - \lambda_1, \tag{2.23}$$

$$F_{x_{m(k)},x_{m(k)-1},a}(\epsilon^{''}) \geq 1 - \lambda_2. \tag{2.24}$$

Again, using (2.21) and by the left continuity property of F we can get

$$F_{x_{m(k)-1},x_{n(k)},a}(\phi(\epsilon_2) - \epsilon^{'} - \epsilon^{''}) \geq 1 - \lambda. \tag{2.25}$$

Now, using (2.23), (2.24), (2.25) in (2.22), we get

$$1-\lambda > \Delta(1-\lambda_1, 1-\lambda_2, 1-\lambda) = 1-\lambda,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

By the construction of the sequence $\{x_n\}$, we have $x_r \in A_1$, $x_{2r} \in A_1$, ..., $x_{nr} \in A_1$. Therefore the subsequence $\{x_{nr}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{nr+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, ..., A_r are closed sets, similarly we get $z \in A_3$, A_4, ..., A_r. Therefore $z \in A_1 \cap A_2 \cap A_3 \cdots \cap A_r$.

Now, we prove that $Tz = z$. For this we have

$$F_{z,Tz,a}(\phi(t)) \geq \Delta(F_{z,Tz,x_n}(s_1), F_{z,x_n,a}(s_2), F_{x_n,Tz,a}(\phi(t)-s_1-s_2)) \tag{2.26}$$

(where $s_1, s_2 > 0$ and $\phi(t) > s_1 + s_2$).

Now, by the property of ϕ, we can choose $\xi_1, \xi_2 > 0$ such that $s_1 = \phi(\xi_1)$ and $\phi(t) - s_1 - s_2 = \phi(\xi_2)$. Now, from (2.26) and using (2.13), we get

$$\begin{aligned} F_{z,Tz,a}(\phi(t)) &\geq \Delta(F_{z,Tz,Tx_{n-1}}(\phi(\xi_1)), F_{z,x_n,a}(s_2), F_{Tx_{n-1},Tz,a}(\phi(\xi_2)) \\ &= \Delta(F_{Tz,Tx_{n-1},z}(\phi(\xi_1)), F_{z,x_n,a}(s_2), F_{Tz,Tx_{n-1},a}(\phi(\xi_2)). \end{aligned}$$

Using (1.3), we get

$$F_{z,Tz,a}(\phi(t)) \geq \Delta(F_{z,x_{n-1},z}\left(\phi\left(\frac{\xi_1}{c}\right)\right), F_{z,x_n,a}(s_2), F_{z,x_{n-1},a}\left(\phi\left(\frac{\xi_2}{c}\right)\right). \tag{2.27}$$

[since $z \in A_{n-1}, x_{n-1} \in A_n$]

Taking limit as $n \to \infty$ in (2.27), by virtue of the properties of ϕ and F, we get

$$F_{z,Tz,a}(\phi(t)) \geq \Delta(1,1,1) = 1.$$

Therefore,

$$F_{z,Tz,a}(\phi(t)) = 1.$$

Hence $z = Tz$, that is, z is a fixed point of T in $A_1 \cap A_2 \cap A_3 \cdots \cap A_r$.

Let v be another fixed point of T, that is, $Tv = v$. Now,

$$\begin{aligned} F_{z,v,a}(\phi(t)) = F_{Tz,Tv,a}(\phi(t)) &\geq F_{z,v,a}\left(\phi\left(\frac{t}{c}\right)\right) \\ = F_{Tz,Tv,a}\left(\phi\left(\frac{t}{c}\right)\right) &\geq F_{z,v,a}\left(\phi\left(\frac{t}{c^2}\right)\right). \end{aligned}$$

Repeating this process n times, we get

$$F_{z,v,a}(\phi(t)) = F_{Tz,Tv,a}(\phi(t)) \geq F_{z,v,a}\left(\phi\left(\frac{t}{c^n}\right)\right).$$

Letting $n \to \infty$ on both sides we get from the above inequality,

$$F_{z,v,a}(\phi(t)) \geq F_{z,v,a}\left(\phi\left(\frac{t}{c^n}\right)\right) \to 1$$

(since ϕ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$). Hence, $F_{z,v,a}(\phi(t)) = 1$, which implies that $z = v$.

Hence T have a unique fixed point in $A_1 \cap A_2 \cap A_3 \cdots \cap A_r$. □

Taking $r = 2$, we get the following Corollary which was established by Choudhury, Das and Bhandari in [10].

Corollary 2.2 [10] *Let (X, F, Δ) be a complete 2-Menger space with the 3rd order minimum t-norm Δ and let there exist two non-empty closed subsets A and B of X such that the mapping $T\colon A \cup B \to A \cup B$ which satisfies the following conditions:*

(i) $TA \subseteq B$ and $TB \subseteq A$

and

(ii) $F_{Tx,Ty,a}(\phi(t)) \geq F_{x,y,a}(\phi(\frac{t}{c}))$

for all $x \in A$, $y \in B$ and $a \in X$ where $0 < c < 1$, ϕ is a ϕ-function. Then $A \cap B$ is non-empty and T has a unique fixed point in $A \cap B$.

The following example satisfied the Theorem 2.2 taking $r = 2$.

Example 2.2 Let $X = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \beta, \delta\}$, $B = \{\gamma, \delta\}$, the t-norm Δ is a 3rd order minimum t-norm and F be defined as

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.40, & \text{if } 0 < t < 4, \\ 1, & \text{if } t \geq 4, \end{cases}$$

$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $T\colon A \cup B \to A \cup B$ as follows: $T\alpha = \delta$, $T\beta = \gamma$, $T\gamma = \delta$, $T\delta = \delta$ then the mapping T satisfies all the conditions of the Theorem 2.2 where $\phi(t) = t$, $0 < c < 1$ and δ is the unique fixed point of T in $A \bigcap B$.

The above example also satisfied Theorem 2.1 where we take the 3rd-order minimum t-norm in place of Hadzic type t-norm.

Remark 2.1 It is to be noted that the method of proof of theorem 2.2 is different from that of Theorem 2.1. This is due to the use of the control function in Theorem 2.2 that the method in the proof of Theorem 2.1 can not be adopted here. Also the Theorem 2.2 could be proved here only with minimum t-norm.

Open problem It remains an open problem whether the proof of Theorem 2.2 can be accomplished with Hadzic type t-norm as in Theorem 2.1. Moreover, the results may possibly by connected with proximity point problems. This will be a new introduction in the context of probabilistic 2-metric spaces.

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