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Boundedness of Third-order Delay Differential Equations in which h is not necessarily Differentiable

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Abstract

In this paper we study the boundedness of solutions of some third-order delay differential equation in which $h(x)$ is not necessarily differentiable but satisfy a Routh–Hurwitz condition in a closed interval $[\delta, kab] \subset (0, ab)$.

Key words: Lyapunov functional, third-order delay differential equation, boundedness

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1 Introduction

This paper studies certain qualitative property of solutions of the delay differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x(t-r)) = p(t, x, \dot{x}, \ddot{x}), \tag{1.1}$$

where a, b and r are positive constants, h and p are continuous functions in their respective arguments.

So far in the literature, much work have been done on the qualitative study (especially, stability and boundedness) of solutions of equation (1.1) (see [18]), as well as of some general equations (see [1],[12]–[17]) using the second (direct) method of Lyapunov ([1]–[18]) by considering Lyapunov functionals and obtaining conditions which ensure the qualitative behavior of solutions of the problem. Often, authors assume h differentiable and make use of the generalized Routh–Hurwitz conditions ([1], [11]–[18]) in one form or the other. The Routh–Hurwitz condition on h, when specialized to the equation (1.1), usually takes the form of restricting $\frac{h(x)}{x}$ $(x \neq 0)$ and/or $h'(x)$ to lie in the (open) "Routh–Hurwitz interval" $(0, ab)$.

In the present work, we discuss the boundedness of solutions of (1.1) in which h is not necessarily differentiable (unlike in [18]), and we shall restrict $\frac{h(x)}{x}$ $(x \neq 0)$ to lie in some special sub-interval of the Routh–Hurwitz interval $(0, ab)$. We shall specifically confine our treatment here to the (closed) sub-interval

$$I_0 \equiv [\delta, kab] \tag{1.2}$$

where $\delta > 0$ is an arbitrary constant and

$$k = \min\left\{\frac{\alpha a(1-\beta)}{2(a+2\alpha)^2}, \frac{\alpha b(1-\beta)}{a(a+\alpha)^2}\right\} < 1 \tag{1.3}$$

with the corresponding Routh–Hurwitz restriction on h taken up in the form

$$\frac{h(\xi)}{\xi} \in I_0 \tag{1.4}$$

for some designated $\xi \neq 0$.

It is not claimed that the value of the constant k given by (1.3) is necessarily the best possible for the result obtained.

2 Preliminary results

Let us give some definitions and a boundedness criterion for the general nonautonomous delay differential system

$$x' = f(t, x_t), \quad x_t = x(t+\theta), \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{2.1}$$

where $f\colon [0,\infty) \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, we assume that f takes bounded sets to bounded sets in $\mathbb{R}^n$. Here $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi\colon [-r, 0] \to \mathbb{R}^n$ with the sup-norm, $r > 0$, and

$$C_H := \{\phi \in C([-r,0], \mathbb{R}^n)\colon \|\phi\| \leq H\}$$

is an open $H-$ ball in C. The standard existence theory [3] implies that if $\phi \in C_H$ and $t \geq 0$, then there exists at least one continuous solution $x(t, t_0, \phi)$ satisfying Eq.(2.1) for $t > t_0$ on $[t_0, t_0+\alpha)$ and such that $x_t(t, \phi) = \phi$, where α is a positive constant. If there exists a closed subset $B \subset C_H$ such that solutions remain in B, then $\alpha = \infty$. In what follows, the symbol $|\cdot|$ stands for the norm in $\mathbb{R}^n$ with

$$|x| = \max_{1 \leq i \leq n} |x_i|$$

Definition 2.1 [3] A continuous strictly increasing function $W\colon [0,\infty) \to [0,\infty)$ such that $W(0) = 0$ and $W(s) > 0$ for $s > 0$, is called a Hahn function. (We denote Hahn functions by W or W_i, where i is an integer.)

Definition 2.2 [3] A function $V\colon [0,\infty)\times D \to [0,\infty)$ is said to be positive definite if $V(t,0)=0$ and there exists a Hahn function W_1 with $V(t,x)\geq W_1(|x|); V$ is said to have an infinitesimal upper limit if there exists a Hahn function W_2 with the condition $V(t,x)\leq W_2(|x|)$.

Definition 2.3 [16] A continuous functional $V\colon [0,\infty)\times C_H \to [0,\infty)$ satisfying a local Lipschitz condition with respect to ϕ is called a Lyapunov functional for Eq.(2.1) if there exists a Hahn function satisfying the following conditions:

(a) $W(|\phi(0)|)\leq V(t,\phi)$ and $V(t,0)=0$;

(b) $\dot{V}_{(2.1)}(t,x_t)=\limsup_{h\to 0}(1/h)[V(t+h,x_{t+h}(t_0,\phi))-V(t,x_t(t_0,\phi))]\leq 0$.

Lemma 2.4 [3] *Let $V\colon [0,\infty)\times C_H\to\mathbb{R}$ be a continuous functional satisfying the local Lipschitz condition. Suppose that the following conditions are satisfied:*

(i) $W(|x(t)|)\leq V(t,x_t)\leq W_1(|x(t)|)+W_2\left(\int_{t-r}^{t}W_3(|x(s)|)ds\right)$;

(ii) $\dot{V}_{(2.1)}\leq -W_3(|x(t)|)+M$ *for some* $M>0$*, where* $W(r)$ *and* W_i $(i=1,2,3)$ *are Hahn functions.*

Then the solutions of Eq.(2.1) are uniformly bounded and uniformly finitely bounded for bound B.

3 Main result

Before we state our result in this section, we write equation(1.1) in the equivalent system form

$$\begin{aligned}&\dot{x}=y, \qquad \dot{y}=z,\\ &\dot{z}=-az-by-h(x)+H(r,x)\int_{t-r}^{t}y(s)ds+p(t,x,y,z),\end{aligned}\tag{3.1}$$

where

$$H(r,x)=\frac{h(x(t))-h(x(t-r))}{x(t)-x(t-r)}.$$

We shall constantly refer to (3.1) subsequently in our discussion.

The following will be our main result.

Theorem 3.1 *Further to the basic assumptions on h and p, assume that the following conditions are satisfied*

(i) (1.4) holds for $\xi\neq 0$;

(ii) $|H(r,x)|\leq L$ *(a positive constant) for all* $x\in\mathbb{R}$;

(iii) $|p(t,x,y,z)|\leq \Delta_0+\Delta_1(|x|+|y|+|z|)$ *for some positive constants* Δ_0 *and* Δ_1 *uniformly in* $t\geq 0$.

Then if Δ_1 is sufficiently small, the solutions of the system (3.1) are uniformly bounded and uniformly ultimately bounded, provided that

$$r<\min\left\{\frac{\delta}{L},\frac{\alpha}{L(1+2\alpha a^{-1})},\frac{2\beta ab}{L\left[2a+2\alpha(1+a^{-1})+b(1-\beta)\right]}\right\}.$$

Proof The main tool in the proof is the Lyapunov functional

$$\begin{aligned}2V(x_t,y_t,z_t) &= \beta(1-\beta)b^2x^2+\beta by^2+2\alpha ba^{-1}y^2+\alpha a^{-1}z^2+\alpha a^{-1}(ay+z)^2\\ &\quad+(z+ay+(1-\beta)bx)^2+\lambda\int_{-r}^{0}\int_{t+s}^{t}y^2(\theta)\,d\theta ds, \qquad (3.2)\end{aligned}$$

where $0<\beta<1$ and $\alpha>0$ are constants.

Obviously, the function $V(x_t,y_t,z_t)$ is positive definite since each term of (3.2) is positive. Hence the condition (i) of Lemma 2.4 is satisfied. Now let us compute the time derivative of the functional $V(x_t,y_t,z_t)$ for the solution (x_t,y_t,z_t) of system (3.1). By $\dot V$, we denote the time derivative of the function $V=V(x_t,y_t,z_t)$ for the solution (x_t,y_t,z_t) of the system (3.1). Then

$$\frac{d}{dt}V(x_t,y_t,z_t)=-U_1-U_2-U_3+U_4+U_5, \qquad (3.3)$$

where

$$\begin{aligned}U_1&=\frac{1}{2}(1-\beta)bh(x)x+\beta aby^2+\frac{1}{2}\alpha z^2\\ U_2&=\frac{1}{4}(1-\beta)bh(x)x+\alpha by^2+(\alpha+a)h(x)y\\ U_3&=\frac{1}{4}(1-\beta)bh(x)x+\frac{1}{2}\alpha z^2+(1+2\alpha a^{-1})h(x)z\\ U_4&=\big((1-\beta)bx+(1+2\alpha a^{-1})z+(\alpha+a)y\big)\,H(r,x)\int_{t-r}^{t}y^2(\theta)d\theta\\ &\quad+\lambda ry^2-\lambda\int_{t-r}^{t}y^2(\theta)d\theta\\ U_5&=\big((1-\beta)bx+(1+2\alpha a^{-1})z+(\alpha+a)y\big)\,p(t,x,y,z).\end{aligned}$$

Next, we derive estimates for some $U_j, j=2,3,4,5$.

There exist positive constants k_1,k_2 such that

$$\begin{aligned}U_2&=\frac{1}{4}\frac{h(x)}{x}\left((1-\beta)b-k_1^{-2}(\alpha+a)\frac{h(x)}{x}\right)x^2+\big(\alpha-k_1^2(\alpha+a)\big)\,y^2\\ &\quad+\left(k_1(\alpha+a)^{\frac{1}{2}}y+2^{-1}k_1^{-1}(\alpha+a)^{\frac{1}{2}}h(x)\right)^2\end{aligned}$$

and

$$\begin{aligned}U_3&=\frac{1}{4}\frac{h(x)}{x}\left((1-\beta)b-k_2^{-2}(1+2\alpha a^{-1})\frac{h(x)}{x}\right)x^2\\ &\quad+\left(\frac{1}{2}\alpha-k_2^2(1+2\alpha a^{-1})\right)z^2+\left(k_2(1+2\alpha a^{-1})^{\frac{1}{2}}z+2^{-1}k_2^{-1}(1+2\alpha a^{-1})^{\frac{1}{2}}h(x)\right)^2\end{aligned}$$

We observe that $U_2\geq 0$ provided

$$\frac{\delta(\alpha+a)}{b(1-\beta)}\leq k_1^2\leq\frac{\alpha b}{\alpha+a},$$

with

$$\delta \leq \frac{h(x)}{x} \leq \frac{\alpha(1-\beta)b^2}{(\alpha+a)^2}. \tag{3.4}$$

Similarly, $U_3 \geq 0$ provided

$$\frac{\delta(1-2\alpha a^{-1})}{b(1-\beta)} \leq k_2^2 \leq \frac{\alpha a}{2(a+2\alpha)},$$

with

$$\delta \leq \frac{h(x)}{x} \leq \frac{\alpha(1-\beta)a^2 b}{2(a+2\alpha)^2}. \tag{3.5}$$

Combining all the inequalities in (3.4) and (3.5), we have for all x, y, z in $\mathbb{R}$,

$$U_j \geq 0 \quad (j = 2, 3) \tag{3.6}$$

if

$$\delta \leq \frac{h(x)}{x} \leq kab \quad \text{with } k = \min\left\{\frac{\alpha(1-\beta)b}{a(\alpha+a)^2}, \frac{\alpha(1-\beta)a}{2(a+2\alpha)^2}\right\} < 1.$$

By condition (ii) of Theorem 3.1, and using $2uv \leq u^2 + v^2$, we have

$$|U_4| \leq \frac{1}{2}(1-\beta)bLrx^2 + \frac{1}{2}(\alpha+a)Lry^2 + \frac{1}{2}(1+2\alpha a^{-1})Lrz^2$$
$$+ \frac{1}{2}L\left((1-\beta)b + (\alpha+a) + (1+2\alpha a^{-1})\right)\int_{t-r}^{t} y^2(\theta)d\theta + \lambda r y^2 - \lambda\int_{t-r}^{t} y^2(\theta)d\theta.$$

If we choose $\lambda = \frac{1}{2}L\left((1-\beta)b + (\alpha+a) + (1+2\alpha a^{-1})\right) > 0$ we must have that

$$\begin{aligned} |U_4| \leq\ & \frac{1}{2}Lr((1-\beta)bx^2 \\ & + (1 + (1-\beta)b + 2(a+\alpha+2\alpha a^{-1}))y^2 + (1+2\alpha a^{-1})z^2). \end{aligned} \tag{3.7}$$

Now, considering U_5, and using condition (iii) of Theorem 3.1 we have that

$$\begin{aligned} |U_5| \leq\ & \left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)\Delta_0 \\ & + \Delta_1\left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)(|x|+|y|+|z|). \end{aligned} \tag{3.8}$$

Combining the estimates (3.6), (3.7) and (3.8) in (3.3), we obtain

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq -\frac{1}{2}(1-\beta)b\left(\frac{h(x)}{x} - Lr\right)x^2$$
$$-\left(\beta ab - \frac{1}{2}Lr\left(2(\alpha+a+2\alpha a^{-1}) + b(1-\beta)\right)\right)y^2$$
$$-\frac{1}{2}\left(\alpha - Lr(1+2\alpha a^{-1})\right)z^2 + \left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)\Delta_0$$
$$+ \Delta_1\left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)(|x|+|y|+|z|).$$

Now, if we choose

$$r < \min\left\{\frac{\delta}{L}, \frac{\alpha}{L(1+2\alpha a^{-1})}, \frac{2\beta ab}{L\left(2a+2\alpha(1+a^{-1})+b(1-\beta)\right)}\right\},$$

we get

$$\begin{aligned}
&\frac{d}{dt}V(x_t, y_t, z_t)\\
&\le -\gamma(x^2+y^2+z^2) + \left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)\Delta_0\\
&\quad + \Delta_1\left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)(|x|+|y|+|z|)\\
&\le -(\gamma-\Delta_1\Delta)(x^2+y^2+z^2) + \left((1-\beta)b|x| + (\alpha+a)|y| + (1+2\alpha a^{-1})|z|\right)\Delta_0,
\end{aligned}$$

where

$$\begin{aligned}
\Delta = \frac{1}{2}\max\big\{&4b(1-\beta)+\alpha+a+1+2\alpha a^{-1}, 4(\alpha+a)+b(1-\beta)+1+2\alpha a^{-1},\\
&\alpha+a+b(1-\beta)+4(1+2\alpha a^{-1})\big\}
\end{aligned}$$

and γ is some positive constant.

If we choose $\Delta_1 < \frac{\gamma}{\Delta}$, then there is some $\theta > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}V(x_t, y_t, z_t) \le -\theta(x^2+y^2+z^2) + n\theta(|x|+|y|+|z|)\\
&= -\frac{\theta}{2}(x^2+y^2+z^2) - \frac{\theta}{2}\left((|x|-n)^2 + (|y|-n)^2 + (|z|-n)^2\right) + \frac{3\theta}{2}n^2\\
&\le -\frac{\theta}{2}(x^2+y^2+z^2) + \frac{3\theta}{2}n^2, \text{ for some } n, \theta > 0.
\end{aligned}$$

Thus condition (ii) of Lemma 2.4 is satisfied by taking

$$W_3(r) = \frac{\theta r^2}{2} \quad \text{and} \quad M = \frac{3\theta n^2}{2}.$$

□

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