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Approximation Spaces in Non-commutative Generalizations of MV-algebras

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Abstract

Generalized MV-algebras (= GMV-algebras) are non-commutative generalizations of MV-algebras. They are an algebraic counterpart of the non-commutative Łukasiewicz infinite valued fuzzy logic. The paper investigates approximation spaces in GMV-algebras based on their normal ideals.

Key words: MV-algebra, GMV-algebra, rough set, approximation space, normal ideal, congruence.

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1 Introduction

Rough sets were introduced by Pawlak [18] in 1982 to give a new mathematical approach to vagueness. The key idea is that our knowledge about the properties of the objects of a given universe of discourse may be inadequate or incomplete in the sense that the objects of this universe can be observed only within the accuracy of indiscernible relations. Rough sets were studied by many authors and from various points of view, see e.g. [1, 3, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17,

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23, 25, 27, 28, 29, 30, 31]. Recall that in the classical rough set theory, subsets are approximated by means of pairs of ordinary sets, so-called lower and upper approximations, which are e.g. composed by some classes of given equivalences.

MV-algebras are an algebraic counterpart of the Łukasiewicz propositional infinite valued logic with truth values from the real interval $[0,1]$. Recently, rough sets in MV-algebras based on their ideals and corresponding congruences were studied in [26].

The first author in [24] and, independently, Georgescu and Iorgulescu in [10], have introduced mutually equivalent generalizations of MV-algebras. We will use for these algebras the name generalized MV-algebras, briefly GMV-algebras. Leuştean in [14] introduced the non-commutative Łukasiewicz infinite valued logic, also with truth values from the real interval $[0,1]$, and she showed that GMV-algebras can be taken as an algebraic semantics of this logic.

In the paper we study approximation spaces in GMV-algebras based on their normal ideals and corresponding congruences. (Note that the quotient GMV-algebras corresponding to normal ideals need not be (commutative) MV-algebras.)

2 Preliminaries

Let $M = (M; \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $\langle 2,1,1,0,0\rangle$. Set $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in M$. Then M is called a *generalized MV-algebra* (briefly: *GMV-algebra*) if for any $x, y, z \in M$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = x = 0 \oplus x$;
(A3) $x \oplus 1 = 1 = 1 \oplus x$;
(A4) $1^- = 0 = 1^\sim$;
(A5) $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;
(A6) $x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x$;
(A7) $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;
(A8) $x^{-\sim} = x$.

The GMV-algebras are in fact equivalent with the pseudo-MV algebras introduced in [10]. The only difference is the following: $x \odot y$ from [24] is $y \odot x$ from [10]. Note that now we use the axiomatization and some basic results from [10] which are modified for the GMV-algebras in this sense.

If we put $x \leq y$ if and only if $x^- \oplus y = 1$, then $\leq$ is an order on M. Moreover, $(M; \leq)$ is a bounded distributive lattice in which $x \vee y = x \oplus (y \odot x^\sim)$ and $x \wedge y = x \odot (y \oplus x^\sim)$ for each x, $y \in M$, and 0 is the least and 1 is the greatest element in M, respectively.

Further we define binary operations d_1 and d_2 (distance functions) on M as follows:

$$d_1(x,y) := (x^- \odot y) \oplus (y^- \odot x), \quad d_2(x,y) := (x \odot y^\sim) \oplus (y \odot x^\sim).$$

Proposition 2.1 ([10]) *The following properties hold in any GMV-algebra:*

(1) $x \odot y = (x^{\sim} \oplus y^{\sim})^{-}$,

(2) $x^{\sim -} = x$,

(3) $0^{\sim} = 0^{-} = 1$,

(4) $x \odot 1 = 1 \odot x = x$,

(5) $(x \oplus y)^{-} = x^{-} \odot y^{-}, \quad (x \oplus y)^{\sim} = x^{\sim} \odot y^{\sim}$,

(6) $(x \odot y)^{-} = x^{-} \oplus y^{-}, \quad (x \odot y)^{\sim} = x^{\sim} \oplus y^{\sim}$,

(7) $x \oplus y = (x^{-} \odot y^{-})^{\sim} = (x^{\sim} \odot y^{\sim})^{-}$,

(8) $x^{-} \oplus x = 1 = x \oplus x^{\sim}$,

(9) $x^{-} \odot x = 0 = x \odot x^{\sim}$,

(10) $x \odot y \leq x \wedge y, \ x \oplus y \geq x \vee y$,

(11) $x \odot (y \odot z) = (x \odot y) \odot z$,

(12) $x \leq y \iff y^{-} \leq x^{-} \iff y^{\sim} \leq x^{\sim}$,

(13) $y \odot (x \oplus z) \leq x \oplus (y \odot z)$,

(14) $(x \oplus z) \odot y \leq (x \odot y) \oplus z$,

(15) $d_1(x, y) = d_1(y, x), \ d_2(x, y) = d_2(y, x)$,

(16) $d_1(x^{\sim}, y^{\sim}) = d_2(x, y), \ d_2(x^{-}, y^{-}) = d_1(x, y)$.

If M is a GMV-algebra and $\emptyset \neq I \subseteq M$, then I is called an *ideal* of M if
(a) $x \oplus y \in I$ for any $x, y \in I$;
(b) $y \leq x$ implies $y \in I$ for any $x \in I$ and $y \in M$.

An ideal I is called *normal* if
(c) $x^{-} \odot y \in I$ iff $y \odot x^{\sim} \in I$ for each $x, y \in M$.

Recall that an ideal I of a GMV-algebra M is normal if and only if $x \oplus I = I \oplus x$, for any $x \in M$, where $x \oplus I := \{x \oplus y \colon y \in I\}$ and $I \oplus x := \{y \oplus x \colon y \in I\}$. It is obvious that the sets $\mathcal{I}(M)$ of ideals of M and $\mathcal{N}(M)$ of normal ideals of M ordered by set inclusion are complete lattices. Moreover, normal ideals of M are in a one-to-one correspondence to congruences on M and the lattice $\mathcal{N}(M)$ is isomorphic to the lattice $Con(M)$ of the congruences on M. (Recall that if I is a normal ideal of M, then the corresponding congruence θ_I is such that $(x, y) \in \theta_I$ iff $d_1(x, y) \in I$ iff $d_2(x, y) \in I$, and if θ is a congruence on M then the corresponding normal ideal I_θ is the 0-class of θ.) If θ is a congruence on M, $I = I_\theta$ the corresponding normal filter of M and $x \in M$ then the class of θ containing x will be denoted by x/θ or x/I.

Now we recall some basic notions of the classical theory of approximation spaces. An *approximation space* is a pair (S, θ) where S is a set and θ an equivalence on S. For any approximation space (S, θ), by the *upper rough approximation* in (S, θ) we will mean the mapping $\overline{Apr}_\theta \colon \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ such that

$$\overline{Apr}_\theta(X) := \{x \in S \colon x/\theta \cap X \neq \emptyset\}$$

and by the *lower rough approximation* in (S, θ) the mapping $\underline{Apr}_\theta \colon \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ such that

$$\underline{Apr}_\theta(X) := \{x \in S \colon x/\theta \subseteq X\},$$

for any $X \subseteq S$. (x/θ is the class of S/θ containing x.) The upper rough approximation $\overline{Apr}_\theta(X)$ of X can be interpreted as the set of all objects which are possibly in X with respect to (S, θ) and the lower rough approximation $\underline{Apr}_\theta(X)$ of X as the set of all objects which are certainly in X with respect to (S, θ).

If $\overline{Apr}_\theta(X) = \underline{Apr}_\theta(X)$ then X is called a *definable* set, otherwise X is called a *rough* set.

The following properties of approximation spaces are well known and obvious.

Proposition 2.2 *If (S, θ) is an approximation space, then for every $X, Y \subseteq S$ we have:*

(1) $\underline{Apr}_\theta(X) \subseteq X \subseteq \overline{Apr}_\theta(X)$.

(2) $X \subseteq Y \Longrightarrow \underline{Apr}_\theta(X) \subseteq \underline{Apr}_\theta(Y), \overline{Apr}_\theta(X) \subseteq \overline{Apr}_\theta(Y)$.

(3) $\overline{Apr}_\theta(X \cup Y) = \overline{Apr}_\theta(X) \cup \overline{Apr}_\theta(Y)$,
$\overline{Apr}_\theta(X \cap Y) \subseteq \overline{Apr}_\theta(X) \cap \overline{Apr}_\theta(Y)$.

(4) $\underline{Apr}_\theta(X \cap Y) = \underline{Apr}_\theta(X) \cap \underline{Apr}_\theta(Y)$,
$\underline{Apr}_\theta(X \cup Y) \supseteq \underline{Apr}_\theta(X) \cup \underline{Apr}_\theta(Y)$.

3 Approximations induced by normal ideals of GMV-algebras

In this section we introduce and investigate special approximation spaces (M, θ) such that M is the universe of a GMV-algebra and θ is a congruence on this GMV-algebra.

Let $M = (M; \oplus, ^-, ^\sim, 0, 1)$ be a GMV-algebra, θ a congruence on M and $I = I_\theta$ the corresponding normal ideal of M. Then for the approximation space (M, θ) we will also use the denotation $\underline{Apr}_I(X)$ for the lower and $\overline{Apr}_I(X)$ for the upper rough approximation, respectively, and any $X \subseteq M$.

Proposition 3.1 *If I is a normal ideal of a GMV-algebra M and $\emptyset \neq X \subseteq M$, then X is definable with respect to I if and only if*

$$\underline{Apr}_I(X) = X \quad \text{or} \quad \overline{Apr}_I(X) = X.$$

Proof Let $\underline{Apr}_I(X) = X$. Let $x \in \overline{Apr}_I(X)$, i.e. $x/I \cap X \neq \emptyset$. Consider $y \in x/I \cap X$. Then $y \in X = \underline{Apr}_I(X)$, hence $x/I = y/I \subseteq X$, therefore $x \in \underline{Apr}_I(X)$.

Further, let $\overline{Apr}_I(X) = X$, $x \in X$ and $y \in x/I$. Then $y/I \cap X \neq \emptyset$, thus $y \in \overline{Apr}_I(X) = X$. □

Remark 3.2 From the proof of Proposition 3.1 it is obvious that the assertion of the preceding proposition is valid for any equivalence on M.

Proposition 3.3 *Let I be a normal ideal of a GMV-algebra M and $a, b \in M$. Then $a/I = b/I$ if and only if there are elements $x, y, u, v \in I$ such that*

$$a = (x \oplus b) \odot y^{\sim} = v^{-} \odot (b \oplus u).$$

Proof If $a/I = b/I$ then $d_1(a, b) \in I$, that means $x = b^{-} \odot a \in I$ and $y = a^{-} \odot b \in I$. Hence $x \oplus b = (b^{-} \odot a) \oplus b = a \vee b = (a^{-} \odot b) \oplus a = y \oplus a$. Further $(x \oplus b) \odot y^{\sim} = (y \oplus a) \odot y^{\sim} = y^{\sim} \wedge a = a$, because $y = a^{-} \odot b \leq a^{-}$, and hence $y^{\sim} \geq a^{-\sim} = a$.

Analogously, if we put $u = a \odot b^{\sim}$ and $v = b \odot a^{\sim}$, then $u, v \in I$ (since $d_2(a, b) \in I$) and $a = v^{-} \odot (b \oplus u)$.

Conversely, let $a = (x \oplus b) \odot y^{\sim}$, where $x,\, y \in I$. Then $((x \oplus b) \odot y^{\sim})^{-} \odot b = ((x \oplus b)^{-} \oplus y^{\sim -}) \odot b = ((x^{-} \odot b^{-}) \oplus y) \odot b \leq ((x^{-} \odot b^{-}) \odot b) \oplus y = (x^{-} \odot (b^{-} \odot b)) \oplus y = y \in I$, therefore $a^{-} \odot b = ((x \oplus b) \odot y^{\sim})^{-} \odot b \in I$.

Further, $b^{-} \odot ((x \oplus b) \odot y^{\sim}) = (b^{-} \odot (x \oplus b)) \odot y^{\sim} \leq (x \oplus (b^{-} \odot b)) \odot y^{\sim} = x \odot y^{\sim} \leq x \in I$, hence $b^{-} \odot a = b^{-} \odot ((x \oplus b) \odot y^{\sim}) \in I$. That means $a/I = b/I$. $\square$

If M is a GMV-algebra and $\emptyset \neq X \subseteq M$, denote by $\langle X \rangle$ the ideal of M generated by X. Obviously $\langle X \rangle = \{a \in M \colon a \leq x_1 \oplus \cdots \oplus x_n$, where $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}\}$. If $\emptyset \neq X, Y \subseteq M$ then $\langle X, Y \rangle$ will denote the ideal $\langle X \cup Y \rangle$.

Theorem 3.4 *Let I be a normal ideal of a GMV-algebra M and $\emptyset \neq X,\, Y \subseteq M$. Then*

$$\overline{Apr}_I(\langle X,\, Y \rangle) \subseteq \langle \overline{Apr}_I(X),\, \overline{Apr}_I(Y) \rangle.$$

If M is linearly ordered then

$$\overline{Apr}_I(\langle X,\, Y \rangle) = \langle \overline{Apr}_I(X),\, \overline{Apr}_I(Y) \rangle.$$

Proof If $a \in \overline{Apr}_I(\langle X, Y \rangle)$ then $a/I \cap \langle X, Y \rangle \neq \emptyset$. Let $b \in a/I \cap \langle X, Y \rangle$ and let $b \leq z_1 \oplus \cdots \oplus z_n$, where $z_i \in X \cup Y$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. Since $a/I = b/I$, there are $c, d \in I$ such that $a = c^{-} \odot (b \oplus d)$. Thus $a = c^{-} \odot (b \oplus d) \leq b \oplus d \leq (z_1 \oplus \cdots \oplus z_n) \oplus d = (z_1 \oplus \cdots \oplus z_{n-1}) \oplus (z_n \oplus d)$, and since $(z_n \oplus d)/I = z_n/I \oplus d/I = z_n/I$ and $z_i \in \overline{Apr}_I(X) \cup \overline{Apr}_I(Y)$, $i = 1, \ldots, n$, we get $a \in \langle \overline{Apr}_I(X), \overline{Apr}_I(Y) \rangle$.

Let M be linearly ordered, $a \in \langle \overline{Apr}_I(X), \overline{Apr}_I(Y) \rangle$, $a \leq v_1 \oplus \cdots \oplus v_n$, $n \in \mathbb{N}$ and $v_i \in \overline{Apr}_I(X) \cup \overline{Apr}_I(Y)$. Let $w_i \in v_i/I \cap X$, whenever $v_i \in \overline{Apr}_I(X)$ and $w_i \in v_i/I \cap Y$, whenever $v_i \in \overline{Apr}_I(Y)$, and let $z \in a/I$. Suppose $a/I \neq (w_1 \oplus \cdots \oplus w_n)/I$. Since M is linearly ordered, $z \leq w_1 \oplus \cdots \oplus w_n$, hence $z \in \langle X, Y \rangle$. Therefore $a \in \overline{Apr}_I(\langle X, Y \rangle)$.

The case $a/I = (w_1 \oplus \cdots \oplus w_n)/I$ is trivial. $\square$

Theorem 3.5 *Let I be a normal ideal of a GMV-algebra M and $\emptyset \neq X, Y \subseteq M$. Then*

$$\langle \underline{Apr}_I(X), \underline{Apr}_I(Y) \rangle \subseteq \underline{Apr}_I(\langle X, Y \rangle).$$

Proof Suppose that $a \in \langle \underline{Apr}_I(X), \underline{Apr}_I(Y)\rangle$ and $z_i \in \underline{Apr}_I(X) \cup \underline{Apr}_I(Y)$, $i = 1, \dots, n$, are such that $a \le z_1 \oplus \cdots \oplus z_n$. Let $b \in a/I$. Then there are $c, d \in I$ such that $b = c^- \odot (a \oplus d)$. Hence $b = c^- \odot (a \oplus d) \le a \oplus d \le (z_1 \oplus \cdots \oplus z_n) \oplus d = (z_1 \oplus \cdots \oplus z_{n-1}) \oplus (z_n \oplus d)$.

We have $z_n \oplus d \in z_n/I \subseteq X$, if $z_n \in \underline{Apr}_I(X)$, and $z_n \oplus d \in z_n/I \subseteq Y$, if $z_n \in \underline{Apr}_I(Y)$, thus $b \in \langle X, Y\rangle$. Therefore $a \in \underline{Apr}_I(\langle X, Y\rangle)$. □

Recall that if I and J are normal ideals of a GMV-algebra M, then $z \oplus I = I \oplus z$ and $z \oplus J = J \oplus z$ for every $z \in M$, and thus $\langle I, J\rangle = \{a \in M \colon a \le x \oplus y,$ where $x \in I,\ y \in J\}$. Moreover, since M satisfies the Riesz Decomposition Property (i.e., if $a, b_1, \dots, b_n \in M$ and $a \le b_1 \oplus \cdots \oplus b_n$, then there are $c_1, \dots, c_n \in M$ such that $c_i \le b_i$, $i = 1, \dots, n$, and $a = c_1 \oplus \cdots \oplus c_n$), $\langle I, J\rangle = I \oplus J := \{x \oplus y \colon x \in I,\ y \in J\}$, $I \oplus J = J \oplus I$, and $\langle I, J\rangle$ is a normal ideal.

Theorem 3.6 *If I and J are normal ideals of a GMV-algebra M and $\emptyset \neq X \subseteq M$, then*

$$\underline{Apr}_{\langle I,J\rangle}(X) \subseteq \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle.$$

Moreover, if $a \in \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$, then

$$a/\langle I, J\rangle \subseteq \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle.$$

Proof Let $a \in \underline{Apr}_{\langle I,J\rangle}(X)$. Then $a/\langle I, J\rangle \subseteq X$, and thus also $a/I \subseteq X$ and $a/J \subseteq X$. Hence $a \le a \oplus a \in \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$, and this means $\underline{Apr}_{\langle I,J\rangle}(X) \subseteq \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$.

Now, let $a \in \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$. Then there are $u_1, \dots, u_n \in \underline{Apr}_I(X) \cup \underline{Apr}_J(X)$ such that $a \le u_1 \oplus \cdots \oplus u_n$.

Let $b \in a/\langle I, J\rangle$. Then there are $x, y \in \langle I, J\rangle$ such that $b = x^- \odot (a \oplus y)$. Moreover $y = v \oplus w$, where $v \in I$ and $w \in J$. Hence $b \le a \oplus y \le u_1 \oplus \cdots u_n \oplus v \oplus w$. We have $(u_n \oplus v)/I = u_n/I$, thus $(u_n \oplus v)/\langle I, J\rangle = u_n/\langle I, J\rangle$. Further $(u_n \oplus v) \oplus w = w_1 \oplus (u_n \oplus v)$, where $w_1 \in J$, and hence $(u_{n-1} \oplus w_1)/J = u_{n-1}/J$, and therefore also $(u_{n-1} \oplus w_1)/\langle I, J\rangle = u_{n-1}/\langle I, J\rangle$. Thus we get $b \in \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$, and this means $a/\langle I, J\rangle \subseteq \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$. □

Theorem 3.7 *If I and J are normal ideals of a GMV-algebra M and X is an ideal of M, then*

$$\underline{Apr}_{\langle I,J\rangle}(X) = \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle.$$

Proof Obviously $\langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle \subseteq X$. Let $a \in \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle$. Then by Theorem 3.6, $a/\langle I, J\rangle \subseteq \langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle \subseteq X$, and so

$$\langle \underline{Apr}_I(X), \underline{Apr}_J(X)\rangle \subseteq \underline{Apr}_{\langle I,J\rangle}(X).$$

□

Theorem 3.8 *If I is a normal ideal and X is a subalgebra of a GMV-algebra M then also $\overline{Apr}_I(X)$ is a subalgebra of M.*

Proof Let $x, y \in \overline{Apr}_I(X)$. Then $x/I \cap X \neq \emptyset \neq y/I \cap X$. Let $x_1 \in x/I \cap X$ and $y_1 \in y/I \cap X$. Then $x_1 \oplus y_1 \in X$ and $x_1 \oplus y_1 \in x/I \oplus y/I = (x \oplus y)/I$ and hence $x \oplus y \in \overline{Apr}_I(X)$.

Now let $x \in \overline{Apr}_I(X)$ and $x_1 \in x/I \cap X$. Then $d_1(x_1, x) \in I$, thus also $d_2(x_1^-, x^-) \in I$, therefore $x_1^- \in x^-/I \cap X$. That means $x^- \in \overline{Apr}_I(X)$.

Analogously one can show that $x^\sim \in \overline{Apr}_I(X)$. □

(The assertion also follows directly from the Third Isomorphism Theorem in the universal algebra [2].)

Theorem 3.9 *Let M be a linearly ordered GMV-algebra, I a normal ideal of M and $X \neq \emptyset$ a convex subset of M. Then $\underline{Apr}_I(X)$ and $\overline{Apr}_I(X)$ are convex too.*

Proof Let $x, y \in \underline{Apr}_I(X)$, $z \in M$ and $x \leq z \leq y$. Let $a \in z/I$ and $x/I \neq z/I \neq y/I$. Since θ_I is also a lattice congruence, and hence has convex classes, for any elements $x_1 \in x/I, y_1 \in y/I, z_1 \in z/I$ we get $x_1 < z_1 < y_1$, thus $z_1 \in \underline{Apr}_I(X)$ and so also $z \in \underline{Apr}_I(X)$. For $x/I = z/I$ or $y/I = z/I$ the proof is obvious. Therefore $\underline{Apr}_I(X)$ is convex.

Let now $x, y \in \overline{Apr}_I(X)$, i.e. $x/I \cap X \neq \emptyset \neq y/I \cap X$. Suppose $z \in M$, $x \leq z \leq y$ and $x_1 \in x/I \cap X$, $y_1 \in y/I \cap X$. If $x/I \neq z/I \neq y/I$ and $z_1 \in z/I$, then $x_1 < z_1 < y_1$. Since x_1, $y_1 \in X$, we get $z_1 \in X$, thus $z_1 \in z/I \cap X$, and hence $z \in \overline{Apr}_I(X)$. Therefore $\overline{Apr}_I(X)$ is convex. □

Remark 3.10 The proof of the preceding theorem is based on the fact that lattice congruences have convex classes. Hence it is easy to show that its assertion is valid for any equivalence with convex classes, e.g. for congruences of algebras with lattice reduct in the signature.

If Y is a subset of a GMV-algebra M, set

$$Y^- := \{y^- : y \in Y\} \quad \text{and} \quad Y^\sim := \{y^\sim : y \in Y\}.$$

Theorem 3.11 *Let I be a normal ideal of a GMV-algebra M and $\emptyset \neq X \subseteq M$. Then*

a) $\overline{Apr}_I(X)^- = \overline{Apr}_I(X^-)$, $\overline{Apr}_I(X)^\sim = \overline{Apr}_I(X^\sim)$;

b) $\underline{Apr}_I(X)^- = \underline{Apr}_I(X^-)$, $\underline{Apr}_I(X)^\sim = \underline{Apr}_I(X^\sim)$.

Proof a) Let $x \in \overline{Apr}_I(X)^-$. Then $x^\sim \in \overline{Apr}_I(X)$, thus $x^\sim/I \cap X \neq \emptyset$. Let $y \in x^\sim/I \cap X$. Then $d_1(x^\sim, y) \in I$, hence also $d_2(x, y^-) \in I$ and $y^- \in X^-$. From this we get $x \in \overline{Apr}_I(X^-)$, i.e. $\overline{Apr}_I(X)^- \subseteq \overline{Apr}_I(X^-)$.

Conversely, let $x \in \overline{Apr}_I(X^-)$, i.e. $x/I \cap X^- \neq \emptyset$. Then there is $y \in x/I \cap X^-$, thus $d_2(x, y) \in I$ and $y^\sim \in X$. From $d_2(x, y) \in I$ we get $d_1(x^\sim, y^\sim) \in I$, hence $x^\sim/I = y^\sim/I$. Thus we have $x^\sim/I \cap X \neq \emptyset$, that means $x^\sim \in \overline{Apr}_I(X)$, and so $x \in \overline{Apr}_I(X)^-$. Hence $\overline{Apr}_I(X^-) \subseteq \overline{Apr}_I(X)^-$.

b) Let $x \in \underline{Apr}_I(X)^-$. Then $x^\sim \in \underline{Apr}_I(X)$, thus $x^\sim/I \subseteq X$, hence $x/I \subseteq X^-$, and that means $x \in \underline{Apr}_I(X^-)$.

Conversely, let $x \in \underline{Apr}_I(X^-)$. Then $x/I \subseteq X^-$, hence $x^\sim/I \subseteq X$, i.e. $x^\sim \in \underline{Apr}_I(X)$, and therefore $x \in \underline{Apr}_I(X)^-$. □

Lemma 3.12 *Let M_1 and M_2 be GMV-algebras, f a homomorphism of M_1 into M_2 and I a normal ideal of M_2. Then $f^{-1}(I)$ is a normal ideal of M_1.*

Theorem 3.13 *Let M_1 and M_2 be GMV-algebras, f a homomorphism of M_1 into M_2, I a normal ideal of M_2 and $\emptyset \neq X \subseteq M_2$. Then*

$$f^{-1}\left(\overline{Apr}_I(X)\right) = \overline{Apr}_{f^{-1}(I)}(f^{-1}(X)).$$

Proof Let $x \in M_1$. Then $x \in \overline{Apr}_{f^{-1}(I)}(f^{-1}(X)) \iff x/f^{-1}(I) \cap f^{-1}(X) \neq \emptyset \iff \exists z \in x/f^{-1}(I) \cap f^{-1}(X) \iff d_1(z,x) \in f^{-1}(I),\ z \in f^{-1}(X) \iff f(d_1(z,x)) \in I,\ z \in f^{-1}(X) \iff f(z)/I = f(x)/I,\ z \in f^{-1}(X)$.

Hence $f(z) \in f(x)/I$ and $z \in f^{-1}(X)$, that means $f(z) \in f(x)/I \cap X$, and so $f(x) \in \overline{Apr}_I(X)$ and this is equivalent to $x \in f^{-1}\left(\overline{Apr}_I(X)\right)$. □

If f is a homomorfismus of a GMV-algebra M_1 into a GMV-algebra M_2, we denote $\mathrm{Ker}(f) = \{x \in M_1 \colon f(x) = 0\}$. It is obvious that $\mathrm{Ker}(f)$ is a normal ideal of M_1.

Theorem 3.14 *Let M_1 and M_2 be GMV-algebras, f a homomorphism of M_1 into M_2 and $\emptyset \neq X \subseteq M_1$. Then*

$$f\left(\overline{Apr}_{\mathrm{Ker}(f)}(X)\right) = f(X).$$

Proof Obviously $f(X) \subseteq f\left(\overline{Apr}_{\mathrm{Ker}(f)}(X)\right)$.

Conversely, $x \in f\left(\overline{Apr}_{\mathrm{Ker}(f)}(X)\right)$ implies that there is $y \in \overline{Apr}_{\mathrm{Ker}(f)}(X)$ such that $x = f(y)$. Let $z \in y/\mathrm{Ker}(f) \cap X$. Then $d_1(z,y) \in \mathrm{Ker}(f)$ and $z \in X$, hence $f(d_1(z,y)) = 0$, thus $d_1(f(z), f(y)) = 0$, and so $f(z) = f(y) = x$, i.e. $f(z) \in f(X)$. Therefore $f\left(\overline{Apr}_{\mathrm{Ker}(f)}(X)\right) \subseteq f(X)$. □

Let M be a GMV-algebra and I a proper ideal of M. Then I is called a *prime ideal* of M if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for any $x, y \in M$. By [10, Theorem 2.17], a normal ideal I of M is prime if and only if the quotient GMV-algebra M/I is linearly ordered. In particular, if M is an MV-algebra, denote by $Spec(M)$ the set of all prime ideals of M. It is well known that for any MV-algebra M, $\bigcap(P \colon P \in Spec(M)) = \{0\}$ [4, Theorem 1.3.3], and hence M is representable as a subdirect product of linearly ordered MV-algebras.

Remark 3.15 Let M be a GMV-algebra and $a, b \in M$. Then by Proposition 3.3, $a/\{0\} = b/\{0\}$ iff $a = (0 \oplus b) \odot 0^\sim = b$. Therefore, if $X \subseteq M$ then $\underline{Apr}_{\{0\}}(X) = X$.

Remark 3.16 Theorem 3.3.1 in [26] asserts that if M is an MV-algebra and $\emptyset \neq X \subseteq M$, then $\bigcap(\underline{Apr}_P(X)\colon P \in Spec(M)) = \{0\}$. But this theorem is not true. Namely, its proof is based only on the inclusion

$$\bigcap(\underline{Apr}_P(X)\colon P \in Spec(M)) \subseteq \underline{Apr}_{\bigcap(P\colon P \in Spec(M))}(X) = \underline{Apr}_{\{0\}}(X) = X.$$

However, this inclusion is trivial since $\underline{Apr}_P(X) \subseteq X$ for any $P \in Spec(M)$, and so it still does not prove the theorem. Moreover, if $0 \notin X$ then consequently $0 \notin \bigcap(\underline{Apr}_P(X)\colon P \in Spec(M))$.

We can illustrate this observation on the following example.

Example 3.17 *Let $M = \{0, a, b, 1\}$ be a four-element Boolean algebra with the least element 0 and the greatest element 1. The prime ideals of M are just $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$. Then e.g.*

$$\begin{aligned}
&\underline{Apr}_{I_1}(\{a, b\}) \cap \underline{Apr}_{I_2}(\{a, b\}) = \emptyset \cap \emptyset = \emptyset,\\
&\underline{Apr}_{I_1}(\{0, a\}) \cap \underline{Apr}_{I_2}(\{0, a\}) = \{0, a\} \cap \emptyset = \emptyset,\\
&\underline{Apr}_{I_1}(\{a, b, 1\}) \cap \underline{Apr}_{I_2}(\{a, b, 1\}) = \{1, b\} \cap \{1, a\} = \{1\}.
\end{aligned}$$

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References

[1] Biswas, R., Nanda, S.: *Rough groups and rough subgroups.* Bull. Polish Acad. Sci., Math. **42** (1994), 251–254.

[2] Burris, S., Sankapanavar, H. P.: A Course in Universal Algebra. *Springer-Verlag*, New York, Heidelberg, Berlin, 1981.

[3] Cataneo, G., Ciucci, D.: *Algebraic structures for rough sets.* Lect. Notes Comput. Sci. **3135** (2004), 208–252.

[4] Cignoli, R. L. O., D'Ottaviano, I. M. L., Mundici, D.: Algebraic Foundation of Many-valued Reasoning. *Kluwer Acad. Publ.*, Dordrecht, Boston, London, 2000.

[5] Ciucci, D.: *On the axioms of residuated structures independence dependencies and rough approximations.* Fundam. Inform. **69** (2006), 359–387.

[6] Ciucci, D.: *A unifying abstract approach for rough models.* In: C. Wang, et al: RSKT 2008, **5009**, *Springer-Verlag*, Heidelberg, 2008, 371–378.

[7] Davvaz, B.: *Roughness in rings.* Inform. Sci. **164** (2004), 147–163.

[8] Davvaz, B.: *Roughness based on fuzzy ideals.* Inform. Sci. **176** (2006), 2417–2437.

[9] Estaji, A. A., Khodaii, S., Bahrami, S.: *On rough set and fuzzy sublattice.* Inform. Sci. **181** (2011), 3981–3994.

[10] Georgescu, G., Iorgulescu, A.: *Pseudo MV-algebras.* Multi. Val. Logic **6** (2001), 95–135.

[11] Kondo, M., M.: *Algebraic approach to generalized rough sets.* In: D. Slezak, et al: RSFD-GrC 2005, LNAI, **5009**, *Springer-Verlag*, Berlin, Heidelberg, 2005, 132–140.

[12] Kuroki, N.: *Rough ideals in semigroups.* Inform. Sci. **100** (1997), 139–163.

[13] Leoreanu-Fotea, V., Davvaz, B.: *Roughness in n-ary hypergroups.* Inform. Sci. **178** (2008), 4114–4124.

[14] Leuştean, I.: *Non-commutative Łukasiewicz propositional logic.* Arch. Math. Logic **45** (2006), 191–213.

[15] Li, T. J., Leung, Y., Zhang, W. X.: *Generalized fuzzy rough approximation operators based on fuzzy coverings.* Int. J. Approx. Reason. **48** (2008), 836–856.

[16] Li, X., Liu, S.: *Matroidal approaches to rough sets via closure operators.* Inter. J. Approx. Reason. **53** (2012), 513–527.

[17] Liu, G. L., Zhu, W.: *The algebraic structures of generalized rough set theory.* Inform. Sci. **178** (2008), 4105–4113.

[18] Pawlak, Z.: *Rough sets.* Inter. J. Inf. Comput. Sci. **11** (1982), 341–356.

[19] Pawlak, Z.: *Rough Sets: Theoretical Aspects of Reasoning about Data.* System Theory, Knowledge Engineering and Problem Solving **9**, *Kluwer Academic Publ.*, Dordrecht, 1991.

[20] Pawlak, Z., Skowron, A.: *Rudiments of rough sets.* Inform. Sci. **177** (2007), 3–27.

[21] Pawlak, Z., Skowron, A.: *Rough sets: some extensions.* Inform. Sci. **177** (2007), 28–40.

[22] Pawlak, Z., Skowron, A.: *Rough sets and Boolean reasoning.* Inform. Sci. **177** (2007), 41–73.

[23] Pei, D.: *On definable concepts of rough set models.* Inform. Sci. **177** (2007), 4230–4239.

[24] Rachůnek, J.: *A non-commutative generalization of MV-algebras.* Czechoslovak Math. J. **52** (2002), 255–273.

[25] Radzikowska, A. M., Kerre, E. E.: *Fuzzy rough sets based on residuated lattices.* In: Transactions on Rough Sets II, Lecture Notes in Computer Sciences, **3135**, *Springer-Verlag*, Berlin, Heidelberg, 2004, 278–296.

[26] Rasouli, S., Davvaz, B.: *Roughness in MV-algebras.* Inform. Sci. **180** (2010), 737–747.

[27] She, Y. H., Wang, G. J.: *An approximatic approach of fuzzy rough sets based on residuated lattices.* Comput. Math. Appl. **58** (2009), 189–201.

[28] Xiao, Q. M., Zhang, Z. L.: *Rough prime ideals and rough fuzzy prime ideals in semigroups.* Inform. Sci. **176** (2006), 725–733.

[29] Yang, L., Xu, L.: *Algebraic aspects of generalized approximation spaces.* Int. J. Approx. Reason. **51** (2009), 151–161.

[30] Zhu, P.: *Covering rough sets based on neighborhoods: An approach without using neighborhoods.* Int. J. Approx. Reason. **52** (2011), 461–472.

[31] Zhu, W.: *Relationship between generalized rough set based on binary relation and covering.* Inform. Sci. **179** (2009), 210–225.