Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **55**, 1 (2016) 87–94

Automorphisms of Spacetime Manifold with Torsion

Vladimir Ivanovich PAN'ZHENSKII [a], Olga Petrovna SURINA [b]

Penza State University,
40 Krasnaya str., Penza, 440026, Russian Federation
[a] *e-mail: kaf-geom@yandex.ru*
[b] *e-mail: pedinstitut@pnzgu.ru*

(Received February 15, 2016)

Abstract

In this paper we prove that the maximum dimension of the Lie group of automorphisms of the Riemann–Cartan 4-dimensional manifold does not exceed 8, and if the Cartan connection is skew-symmetric or semisymmetric, the maximum dimension is equal to 7. In addition, in the case of the Riemann–Cartan n-dimensional manifolds with semisymmetric connection the maximum dimension of the Lie group of automorphisms is equal to $n(n-1)/2+1$ for any $n > 2$.

Key words: Riemann–Cartan manifolds, automorphisms, semi-symmetric connection.

2010 Mathematics Subject Classification: 53C05, 53C25

As is known, the development of the general theory of relativity has led to the first geometrization of the gravitational field. The geometry of space-time in the general theory of relativity is that of a four-dimensional pseudo-Riemannian manifold of Lorentz signature. Then, the problem of geometrizing the unified theory of gravitation and electromagnetism was posed. It was from solving this problem 1 that the new non-Riemannian geometry arose. The first such geometry was proposed by Weyl in 1918. Another version of geometrization of gravitation and electromagnetism was proposed by E. Cartan in 1922 (see, e.g., the survey [1]). In the geometry of Cartan, the Levi-Civita connection is replaced by a metric connection with torsion. As a result, the space-time manifold is endowed with both curvature and torsion. In the sequel, this approach has led to the development of Einstein–Cartan theory. In this theory, the Cartan connection is assumed to be semisymmetric and the trace of its torsion tensor is identified, up to a dimensional multiplier, with the vector potential of the electromagnetic field. Many versions of geometrization of physical theories unifying different forms of interactions inevitably lead to accounting torsion; multidimensional spaces with positive-definite metric are also used as model spaces (e.g., in quantum field theory [2, 3]).

In papers [4, 5, 6, 7] it is determined that the dimension of the Lie group of automorphisms of the Riemann-Cartan n-dimensional manifold is less than $\frac{n(n+1)}{2}$ with $n \neq 3$ and is equal to 6, if $n = 3$. If metric g is positively defined and $n > 4$, maximum dimension is equal to $\frac{n(n-1)}{2} + 1$. The case $n = 4$ is exceptional, as the orthogonal group $SO(n)$ when $n = 4$ is not simple (it is semisimple) [8].

Next, we prove that in the case $n = 4$, the dimension of the Lie group of automorphisms does not exceed 8, and if the connection is skew-symmetric or semisymmetric or is exactly equal to 7. In addition, for semisymmetric connection the dimension of the Lie group of automorphisms is exactly equal to any $n > 2$. It being known that the metric tensor may have any signature

A smooth n-dimensional manifold M with a semi-Riemannian metric g and a linear metric connection $\tilde{\nabla}$ with torsion is called a Riemann–Cartan manifold [1]. The connection $\tilde{\nabla}$ can be represented as

$$\tilde{\nabla} = \hat{\nabla} + \frac{1}{2}\tilde{S},$$

where $\hat{\nabla}$ is an associated symmetric connection and $\tilde{S}$ is a torsion tensor of the connection $\tilde{\nabla}$. On the other hand, $\tilde{\nabla} = \nabla + \tilde{T}$, where ∇ is the Levi-Civita connection of the metric g and $\tilde{T}$ is the deformation tensor of the connection ∇. The covariant deformation tensor T determined by the equality

$$T(X, Y, Z) = g(\tilde{T}(X, Y), Z)$$

is skew-symmetric with respect to the last two arguments because of the covariant constancy of the metric tensor g in the connection $\tilde{\nabla}$, $\tilde{\nabla} g = 0$. Thus, the Riemann–Cartan structure $(g, \tilde{\nabla})$ is unambiguously defined by setting a pair of tensor fields (g, T), namely, a metric tensor and a deformation tensor, the first of which is symmetric with respect to its arguments, and the second one is skew-symmetric in the last two arguments. We also note that the deformation tensor is defined unambiguously by the torsion tensor and vice versa, while the symmetric part $\hat{\nabla}$ of the connection $\tilde{\nabla}$ coincides with the Levi–Civita connection ∇ if and only if the tensor T is skew-symmetric with respect to its arguments [8]. In this case, $T = \frac{1}{2}S$, where S is a covariant tensor of torsion, and the connection $\tilde{\nabla}$ is called skew-symmetric. If

$$S(X, Y, Z) = g(X, Z)\Theta(Y) - g(Y, Z)\Theta(X),$$

where $\Theta = \frac{1}{n-1} \operatorname{trace} \tilde{S}$, then the connection $\tilde{\nabla}$ is called semi-symmetric.

A diffeomorphism $\varphi \colon M \to M$ is said to be an automorphism of the Riemann–Cartan manifold if g and $\tilde{\nabla}$ remain invariant under φ. Since $\tilde{\nabla} = \nabla + \tilde{T}$ and the invariance of ∇ follows from the invariance of g [3], then the connection $\tilde{\nabla}$

is invariant if and only if the deformation tensor $\tilde{T}$ is invariant, which is equivalent to the invariance of the covariant deformation tensor T. Thus the set of all automorphisms of the Riemann–Cartan manifold $(M, g, \tilde{\nabla})$ either coincides with the Lie group of isometrics of semi-Riemannian manifold (M, g) or is its closed Lie subgroup which leaves the tensor field T invariant and, therefore, it has the dimension $r \leq \frac{n(n+1)}{2}$.

Theorem 1 *The dimension of the group of automorphisms of the Riemann–Cartan four-dimensional manifolds does not exceed* 8.

Proof Let G be a group of automorphisms of the Riemann–Cartan M. The stationary subgroup of $x_0 \in M$ induces isotropy group G_0 in tangent space $E = T_{x_0}M$, which is a subgroup of (pseudo) orthogonal transformations of (pseudo) Euclidean vector space $E = E^4_{p,q}$. Since the strain tensor field T is invariant as to G, then the value of this field at point x_0 is a nonzero tensor on E, invariant as to G_0. Let us consider T as linear display $E \cdot E \cdot E \to R$. Let ξ be a Lie algebra element of the Lie group of (pseudo) orthogonal transformations in space E and φ_t be a single parameter group of transformations, generated by ξ. Then ξ, belongs to the Lie algebra g_0 of the Lie group G_0, if and only if φ_t leaves tensor invariant, i.e.

$$T(\varphi_t u, \varphi_t v, \varphi_t w) = T(u, v, w), \quad u, v, w \in E, \; t \in R$$

Differentiating this equation as to t, at $t = 0$, we obtain that $\xi \in g_0$ when and only when

$$T(\xi u, v, w) + T(u, \xi v, w) + T(u, v, \xi w) = 0.$$

Let $(e_1, \dots, e_n)$ be orthonormal basis in E, T_{ijk} and components T and ξ in this basis. In coordinates this equation takes the form:

$$(T_{pjk}g_{is} + T_{ipk}g_{js} + T_{ijp}g_{ks})\xi^{ps} = 0,$$

where $\xi^{ps} = \xi^s_q g^{qp}$, $g_{is}g^{sj} = \delta^j_i$, δ^j_i — Kronecker symbol.

The algebra of the Lie group of (pseudo) orthogonal transformations consists of the matrix in the following form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

Where A is a skew-symmetric matrix of the size $p \times p$, D is a skew-symmetric matrix of the size $q \times q$, $C = B^T$—matrix transposed to B. Therefore $\xi^1_1 = \xi^2_2 = \ldots = \xi^n_n = 0$, a $\xi^j_i = \pm\xi^{ps}$. Taking into account the Lie algebra structure, it is not difficult to find out that the minimum number of linear independent equations of the system for $n = 4$ is equal to 2. Indeed, let T_{ijk} is one of nonzero components of tensor T. Since T_{ijk} is skew symmetric with respect to j and k, then $j \neq k$. Having renumbered the basis if necessary, we can assume that either $i = 1$, $j = 2$, $k = 3$, or $i = j = 1$, $k = 2$, i.e. $T_{123} \neq 0$ or $T_{112} \neq 0$.

Consider the first case. Then the equations of the system (2), numbered by indices $i = 1$, $j = 2$, $k = 4$ and $i = 1$, $j = 3$, $k = 4$ have the form

$$\dots 0 \cdot \xi^{24} \pm T_{123}\xi^{24} \neq 0$$

$$\dots \pm T_{123}\xi^{24} + 0 \cdot \xi^{24} \neq 0$$

and, consequently, they are linearly independent. If $T_{112} \neq 0$, then we have two linearly independent equations, numbered by indices $i = j = 1$, $k = 3$ and $i = j = 1$, $k = 4$:

$$\dots \pm T_{112}\xi^{23} + 0 \cdot \xi^{24} \neq 0$$

$$\dots 0 \cdot \xi^{23} \pm T_{112}\xi^{24} \neq 0.$$

Since the rank of the system is at least two, then the dimension of isotropy group, G_0 is not more than four, and the dimension of the whole automorphisms group is not more than 8. □

Connection $\tilde{\nabla}$ is called skew-symmetric if strain tensor T is skew symmetric regarding its arguments. In this case, T_{ijk}, components comprising two identical indexes are equal to zero. Taking this fact into account and considering that, for example, $T_{123} \neq 0$, we can point to three linearly independent equations of the system. Indeed, consider the subsystem of the system, the equations of which are numbered by the following indices: $i = 1$, $j = 2$, $k = 4$; $i = 1$, $j = 3$, $k = 4$; $i = 2$, $j = 3$, $k = 4$. The matrix consisting of columns of the subsystem at unknown ξ^{ps} with indices $(1, 4)$, $(2, 4)$ and $(3, 4)$ has the form

$$\begin{pmatrix} 0 & 0 & \pm T_{123} \\ 0 & \pm T_{123} & 0 \\ \pm T_{123} & 0 & 0 \end{pmatrix}$$

And it is obviously non-degenerate. Thus we have

Theorem 2 *The dimension of the Lie group of automorphisms of the Riemann–Cartan four-dimensional manifolds with skew-symmetric connection does not exceed* 7*.*

Theorem 3 *The dimension of the Lie group of automorphisms of an n-dimensional Riemann–Cartan manifold with semi-symmetric connection is not larger than* $\frac{n(n-1)}{2} + 1$.

Proof Let G be an r-dimensional Lie group of automorphisms of an n-dimensional Riemann–Cartan manifold M. The stationary subgroup of a point $x_0 \in M$ induces the isotropy group G_0 in the tangent space $E = T_{x_0}M$. The vector space $E = E^n_{p,q}$ is an n-dimensional Euclidean space ($p = n$, $q = 0$) or a semi-Euclidean space with (p, g)-signature $(+, \dots, +, -, \dots, -)$. The value of a torsion tensor field $\tilde{S}$ at $x_0 \in M$ is a nonzero tensor in $(E^* \wedge E^*) \otimes E$. Let us consider $\tilde{S}$ as a skew-symmetric mapping $E \times E \to E$. The isotropy group G_0 is a subgroup of the group of orthogonal or pseudo-orthogonal transformations of E. Let ξ be an element of the Lie algebra of the Lie group of (pseudo) orthogonal transformations of E, and $\varphi_t = \exp t\xi$, be a one-parameter subgroup of transformations generated by ξ. Then ξ belongs to the Lie algebra go of the Lie group G_0 if and only if the tensor S remains invariant under φ_t, i.e.,

$$\tilde{S}(\varphi_t u, \varphi_t v) = \varphi_t \tilde{S}(u, v). \tag{1}$$

Differentiating (1) with respect to t at $t = 0$, we get

$$\tilde{S}(\xi u, v) + \tilde{S}(u, \xi u) = \xi \tilde{S}(u, v). \tag{2}$$

Let $(e_1, \dots, e_n)$ be a (pseudo) orthonormal basis in E, and S^k_{ij} and ξ^j_i be components $\tilde{S}$ and ξ in this basis. Then (2) has the form

$$S^k_{sj}\xi^s_i + S^k_{is}\xi^s_j - S^r_{ij}\xi^k_r = 0 \tag{3}$$

or

$$(S^k_{sj}\delta^r_i + S^k_{is}\delta^s_j - S^r_{ij}\delta^k_s)\xi^s_r = 0, \tag{4}$$

where δ^j_i is the Kronecker symbol.

Let now the connection $\tilde{\nabla}$ be semi-symmetric. Then

$$S^k_{ij} = \frac{1}{n-1}(\delta^k_i \eta_j - \delta^k_j \eta_i). \tag{5}$$

where $\eta_j = S^*_{*j}$ denote the components of a 1-form in E. Substituting (5) into (4), we get

$$(\delta^k_s \eta_j \delta^r_i - \delta^k_j \eta_s \delta^r_i + \delta^k_i \eta_s \delta^r_i - \delta^k_s \eta_i \delta^r_j - \delta^r_i \eta_j \delta^k_s + \delta^r_j \eta_i \delta^k_s)\xi^s_r = 0$$

or

$$(\delta^k_i \delta^r_j - \delta^k_j \delta^r_i)\eta_s \xi^s_r = 0. \tag{6}$$

Let us prove that this system contains at least $n-1$ linearly independent equations. Actually, (6) can be written as follows:

$$(\delta^k_i \delta^r_j - \delta^k_j \delta^r_i)\eta_s \xi^s_r + (\delta^k_i \delta^s_j - \delta^k_j \delta^s_i)\eta_r \xi^r_s = 0. \tag{7}$$

As the 1-form η is nonzero, then at least one of its coordinates is not zero. Let $\eta_s \neq 0$ for some s. We consider the subsystem consisting of $n-1$ equations with the indices $i = k = s$, $j = 1, \dots, n$; $j \neq s$. The subsystem takes the form

$$\dots + (\delta^s_s \delta^r_j - \delta^s_j \delta^r_s)\eta_s \xi^s_r + (\delta^s_s \delta^s_j - \delta^s_j \delta^s_s)\eta_r \xi^r_s + \dots = 0$$

or

$$\dots + \delta^r_j \eta_s \xi^s_r + \dots = 0 (r = 1, \dots, n; r \neq s), \tag{8}$$

and it is linearly independent because the matrix $(\delta^r_j \eta_s)$ is obviously nondegenerate. Therefore the dimension of the isotropy group G_0 is not larger than $\frac{n^2-n}{2} - (n-1)$ and the dimension of the group of all automorphisms is not larger than $\frac{n^2-n}{2} - (n-1) + n = \frac{n(n-1)}{2} + 1$. $\square$

Theorem 4 *The maximum dimension of the Lie group of automorphisms of an n-dimensional Riemann–Cartan manifold with semi-symmetric, connection is equal to* $\frac{n(n-1)}{2}+1$.

Proof To prove the theorem, it is enough to give an example of an n-dimensional Riemann–Cartan manifold with automorphism group of dimension $\frac{n(n-1)}{2}+1$. Let us consider the semi-Riemannian space M^n, $n>3$, with the metric form

$$ds^2 = {dx^1}^2 + e^{2Hx^1}(\varepsilon_2 {dx^2}^2 + \ldots + \varepsilon_n {dx^n}^2), \tag{9}$$

where $\varepsilon_\alpha = \pm 1$, $\alpha = 2,\ldots,n$, $H=\text{const}$. Calculating the curvature tensor of this space, we verify the validity of the equality

$$R_{ijkl} = -H^2(g_{il}g_{jk} - g_{ik}g_{jl}).$$

It follow then that M^n has a constant sectional curvature $k=-H^2$. Consequently, the isometry group of this space has the maximum dimension $\frac{n(n+1)}{2}$. Let us consider a closed subgroup of the group containing all isometries which leave invariant a single vector field orthogonal to the semi-Euclidean subspace E^{n-1}, $x^1=\text{const.}$, with the metric form

$$d\sigma^2 = \varepsilon_2 {dx^2}^2 + \ldots + \varepsilon_n {dx^n}^2. \tag{10}$$

Basic operators of this subgroup are

$$\partial_\alpha, -\varepsilon_\alpha x^\beta \partial_\alpha + \varepsilon_\beta x^\alpha \partial_\beta, -\frac{1}{H}\partial_1 + x^\alpha \partial_\alpha, \quad \alpha < 0,\ \alpha,\beta = 2,\ldots,n. \tag{11}$$

In (11), the first $\frac{n(n-1)}{2}$ vector fields are basic operators of the Lie group of isometries of the space E^{n-1} with metric (10), and the last vector field is defined by the invariance of the metrics (9) and a single vector field orthogonal to E^{n-1} with respect to the last vector field. The condition of the invariance of the deformation tensor T_{ijk} with respect to the vector field $X=\xi^p\partial_p$ takes the form

$$\xi^p\partial_p T_{ijk} + \partial_i\xi^p T_{pjk} + \partial_j\xi^p T_{ipk} + \partial_k\xi^p T_{ijp} = 0. \tag{12}$$

To find the deformation tensor T_{ijk}, which is invariant with respect to the group of operators (11), it is necessary to write a corresponding differential equation (12) for each vector field (11) and then integrate the obtained system of partial differential equations. Fortunately, this task becomes much more simplified if the connection is semi-symmetric. For the semi-symmetric connection we have

$$T_{ijk} = \frac{1}{n-1}(g_{ik}\eta_j - g_{ij}\eta_k). \tag{13}$$

Hence the invariance of T_{ijk} leads to the invariance of $\eta_j = T^*_{*j}$ and vice versa. That is wiry, it is enough to integrate the equations of the invariance of η

$$\xi^p\partial_p\eta_j + \partial_j\xi^p\eta_p = 0 \tag{14}$$

and restore T_{ijk} according to (13). As a result, we get $\eta = c\,dx^1$, $c = \text{const.}$, and

$$T = ae^{2Hx^1} \sum_{\alpha} \varepsilon_\alpha \, dx^\alpha \otimes dx^\alpha \wedge dx^1, \quad a = \text{const.} \tag{15}$$

Thus we have an example of the n-dimensional Riemann–Cartan manifold ($n > 3$) with semi-symmetric connection whose automorphism group has dimension $\frac{n(n-1)}{2} + 1$. The metric tensor and deformation tensor of the manifold are defined by (9) and (15), respectively, and basic operators arc defined by (11). □

In Einstein's general theory of relativity (GTR) and its generalizations, the basic subject is a four-dimensional semi-Riemannian manifold of signature $(+---)$ which we call a space-time manifold M^4. The metric form for M^4, given in the proof of Theorem 2, can be rewritten in the following way:

$$ds^2 = {dx^0}^2 - e^{2Hx^0}({dx^1}^2 + {dx^2}^2 + {dx^3}^2), \quad x^0 = ct. \tag{16}$$

It is well known (see, e.g., [4]) that this metric is the solution of the Einstein equation with Λ-term

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij} + \Lambda g_{ij},$$

which defines the stationary model of the Universe. Contrary to Friedmann's solutions, this solution has no singularity. Nevertheless, the metrics of the stationary model describes an expansion of the Universe occurring without a bound in time both in the past and in the future. The Hubble constant H (redshift of spectral lines) is then unchanged during the Universe evolution, and the cosmological constant is $\Lambda = \frac{3H}{c^2}$. A spatial section ($x^0 = ct = \text{const.}$) is a Euclidean space, i.e., in this model the world is flat and without matter, therefore the theory of the stationary Universe cannot he applied to the Universe with matter. But endowing a stationary model with additional structures, as is done, for example, in the theory of compensations, may allow solving some problems existing in the framework of the theory of a stationary Universe. Cartan was the first to draw the physicists' attention to the need that torsion be taken into account for generalizations of GTR ((1922), see, e.g., [9]). In one of the attempts to create the uniform gravitation and electromagnetic theory (1928), Einstein used a connection with torsion but without curvature (connection of absolute parallelism). Subsequently in the Einstein–Cartan theory, the torsion is introduced to geometrize the matter spin density, the spin is represented by a covector η defining the torsion. It means that the connection $\tilde{\nabla}$ must be semi-symmetric. Moreover, for this connection to have the maximum symmetry number, the deformation tensor of the connection must have the following form according to formula (15):

$$T = ae^{2Hx^0} \sum_{\alpha=1}^{3} dx^\alpha \otimes dx^\alpha \wedge dx^0, \quad a = \text{const.} \tag{17}$$

By integrating the invariance equation of deformation tensor (12) with respect to the isometry group (11), we obtain the general solution

$$T = ae^{2Hx^0} \sum_{\alpha=1}^{3} dx^\alpha \otimes dx^\alpha \wedge dx^0 + be^{3Hx^0} dx^1 \wedge dx^2 \wedge dx^3, \tag{18}$$

where $a, b = \text{const}$. Alongside with the "spin" part T_α, there is the skew-symmetric part T_a, defining the torsion of the spatial section $x^0 = \text{const.}$ which may not be a spin.

References

[1] Gordeeva, I. A., Pan'zhenskii, V. I., Stepanov, S. E.: *Riemann–Cartan manifolds.* In: Modern Mathematics and Its Applications **123** Geometry, *VINITI*, Moscow, 2009, 110–141, (in Russian).

[2] Tamm, I. E.: *On the curved momentum space.* In: Selected Papers **4**, *Springer-Verlag*, Berlin, Heidelberg, 1991, 197–210; Selected Scientific Papers **2**, Nauka, Moscow, 1975, 218–225.

[3] Tamm, I. E., Vologodskii, V. G.: *On the use of curved momentum space in the construction of nonlocal Euclidean field theory.* In: Collection of Scientific Papers **2**, *Nauka*, Moscow, 1975, 226–253, (in Russian).

Princeton Univ. Press, Princetton, NJ, 1953; Inostrannaya Literatura, Moscow, 1957.

[4] Pan'zhenskii, V. I.: *Maximally movable Riemannian spaces with torsion.* Math. Notes **85**, 5-6 (2010), 720–723; Mat. Zametki 85, 5 (2009), 754–757.

[5] Pan'zhenskii, V. I.: *Automorphisms of the Riemann–Cartan space-time manifold.* Tr. Mezhdunar. Geom. Tsentra **5**, 2 (2012), 27–34.

[6] Pan'zhenskii, V. I.: *Automorphisms of Riemann-Cartan Manifolds with Semi-Symmetric Connection.* Journal of Mathematical Physics, Analysis, Geometry **10**, 2 (2014), 233–239.

[7] Pan'zhenskii, V. I.: *Automorphisms of Riemann–Cartan Manifolds.* Math. Notes **98**, 4 (2015), 613–623.

[8] Yano, K., Bochner, S.: Curvature and Betti Numbers. *Princeton Univ. Press*, Princetton, NJ, 1953; Inostrannaya Literatura, Moscow, 1957.

[9] Stepanov, S. E., Gordeeva, I. A.: *Pseudo-Killing and pseudoharmonic vector Fields on a Riemann–Cartan manifold.* Math. Notes **87**, 1-2 (2010), 248–257; Mat. Zametki 87, 2 (2010), 267–279.