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Nonlinear Implicit Hadamard's Fractional Differential Equations with Delay in Banach Space

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Abstract

In this paper, we establish sufficient conditions for the existence of solutions for nonlinear Hadamard-type implicit fractional differential equations with finite delay. The proof of the main results is based on the measure of noncompactness and the Darbo's and Mönch's fixed point theorems. An example is included to show the applicability of our results.

Key words: Hadamard's fractional derivative, implicit fractional differential equations in Banach space, fractional integral, existence, Gronwall's lemma for singular kernels, Measure of noncompactness, fixed point.

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1 Introduction

Fractional calculus has evolved into an important and interesting field of research in view of its numerous applications in technical and applied sciences.

The mathematical modeling of many real world phenomena based on fractional-order operators is regarded as relevant and different than the one depending on integer-order operators. In particular, fractional calculus has played a significant role in the recent development of special functions and integral transforms, signal processing, control theory, bioengineering and biomedical, viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, etc. For further details and applications, see the monographs [1, 2, 8, 17, 22, 24]. Fractional differential equations involving Riemann–Liouville and Caputo type fractional derivatives have extensively been studied by many researchers such as, for example, in [11, 12, 21]. However, the literature on Hadamard type fractional differential equations is not enriched yet. The fractional derivative due to Hadamard, introduced in 1892 ([16]), differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral and their applications to differential equations can be found in [4, 5, 13, 17, 18]. For some recent work on the topic, we refer to [23, 26] and the references cited therein.

In this paper, we establish existence, uniqueness results to the following nonlinear implicit fractional differential equation with finite delay

$$D^\nu y(t) = f(t, y_t, D^\nu y(t)), \quad \text{for each } t \in J = [1, T],\ 0 < \nu \leq 1, \tag{1}$$

$$y(t) = \varphi(t), \quad t \in [1-r, 1],\ r > 0, \tag{2}$$

where D^ν is the Hadamard fractional derivative, $(E, \|\cdot\|)$ is a real Banach space, $f\colon J \times C([-r,0],E) \times E \to E$ is a given function, $\varphi \in C([1-r,1],E)$ with $\varphi(1) = 0$. For each function y_t defined on $[1-r,T]$ and for any $t \in J$, we denote by y_t the element of $C([-r,0],E)$ defined by:

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r, 0],$$

$y_t(.)$ represents the history of the state from time $t-r$ up to time t.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some preliminaries about fractional calculus and the Kuratowski's measure of noncompactness and auxiliary results. In Section 3, two results are discussed; the first one is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, the second one is based on Mönch's fixed point theorem. In the last section, we present an example illustrating the presented main results.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(E, \|\cdot\|)$ be a Banach space. We denote by $C(J,E)$ the space of E-valued continuous functions on J with the usual supremum norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\} \quad \text{for any } y \in C(J,E).$$

Also $C([-r,0],E)$ is endowed with the norm

$$\|y\|_C = \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\}.$$

A measurable function $y\colon J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

Let $L^1(J,E)$ denote the Banach space of measurable functions $y\colon J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_1^T \|y(t)\|\,dt.$$

For properties of the Bochner integrable, see [25].

Definition 2.1 ([17]) The Hadamard fractional (arbitrary) order integral of the function $h \in L^1(J,E)$ of order $\nu \in \mathbb{R}_+$ is defined by

$$I^\nu h(t) = \frac{1}{\Gamma(\nu)} \int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} \frac{h(s)}{s}\,ds,$$

where Γ is the Euler gamma function defined by

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t}\,dt, \quad \nu > 0$$

and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 ([17]) For a function $h\colon [1,\infty) \to E$, the Hadamard fractional-order derivative of order ν of h, is defined by

$$D^\nu h(t) = \frac{1}{\Gamma(n-\nu)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\nu-1} \frac{h(s)}{s}\,ds,$$

where $n = [\nu]+1$ and $[\nu]$ denotes the integer part of the real number ν.

Corollary 2.3 ([17]) *Let $\nu > 0$ and $n = [\nu]+1$. The equality $D^\nu h(t) = 0$ is valid if and only if*

$$h(t) = \sum_{j=1}^n c_j \left(\log t\right)^{\nu-j} \quad \textit{for each } t \in J,$$

where $c_j \in \mathbb{R}$ $(j = 1,\ldots,n)$ are arbitrary constants.

In particular, when $0 < \nu \leq 1$, the relation $D^\nu h(t) = 0$ holds if, and only if

$$h(t) = c(\log t)^{\nu-1} \quad \textit{for any } c \in \mathbb{R}.$$

Moreover, for a given set V of functions $v\colon J \to E$ let us denote by

$$V(t) = \{v(t), v \in V\}, \quad t \in J$$

and

$$V(J) = \{v(t) \colon v \in V, \quad t \in J\}.$$

Next we give the definition of the concept of a measure of noncompactness and some auxiliary result; see for more details [6, 9, 10] and the references therein.

Definition 2.4 ([9]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha \colon \Omega_E \to [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } \operatorname{diam}(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where $\operatorname{diam}(B_i) = \sup\{||x - y|| : x, y \in B_i\}$.

The Kuratowski measure of noncompactness satisfies the following properties.

Lemma 2.5 ([6, 9, 10]) *Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.*

(b) nonsingularity: α is equal to zero on every one element-set.

(c) $\alpha(B) = \alpha(\overline{B}) = \alpha(\operatorname{conv} B)$, *where* $\operatorname{conv} B$ *is the convex hull of B.*

(d) monotonicity: $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(e) algebraic semi-additivity: $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where*

$$A + B = \{x + y : x \in A,\ y \in B\}.$$

(f) semi-homogencity: $\alpha(\lambda B) = |\lambda| \alpha(B)$; $\lambda \in \mathbb{R}$, *where* $\lambda B = \{\lambda x : x \in B\}$.

(g) semi-additivity: $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(h) invariance under translations: $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in E$.

For our purpose we will only need the followings fixed point theorems, and the important Lemma.

Theorem 2.6 (Darbo's Fixed Point Theorem) ([14]) *Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N \colon C \to C$ is such that for all closed subsets D of C,*

$$\alpha(N(D)) \leq k\alpha(D), \tag{3}$$

where $0 \leq k < 1$. Then N has a fixed point in C.

Remark 2.7 Mappings satisfying the Darbo-condition (3) have subsequently been called *k-set contractions.*

Theorem 2.8 (Mönch's Fixed Point Theorem) ([3, 20]) *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{\text{conv}} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.9 ([15]) *If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \alpha(V(t))$ is continuous on J, and

$$\alpha_c(V) = \sup_{1 \le t \le T} \alpha(V(t)).$$

(ii)

$$\alpha\left(\int_1^T x(s)\, ds : x \in V\right) \le \int_1^T \alpha(V(s))\, ds,$$

where $V(s) = \{x(s) : x \in V\}$, $s \in J$.

Theorem 2.10 (Ascoli–Arzela) ([14]) *Let $A \subset C(J, E)$, A is relatively compact (i.e. $\overline{A}$ is compact) if:*

1. *A is uniformly bounded i.e., there exists $M > 0$ such that*

$$\|f(t)\| < M \text{ for every } f \in A \text{ and } t \in J.$$

2. *A is equicontinuous i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $t, \overline{t} \in J$, $\left|t - \overline{t}\right| \le \delta$ implies $\|f(t) - f(\overline{t})\| \le \epsilon$, for every $f \in A$.*
3. *The set $\{f(t) \colon f \in A; t \in J\}$ is relatively compact in E.*

Lemma 2.11 ([19]) *Let $\upsilon \colon [1, T] \longrightarrow [0, +\infty)$ be a real function and $\omega(\cdot)$ is a nonnegative, locally integrable function on $[1, T]$ and there are constants $a > 0$ and $0 < \alpha \le 1$ such that*

$$\upsilon(t) \le \omega(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \frac{\upsilon(s)}{s}\, ds,$$

then

$$\upsilon(t) \le \omega(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \frac{\omega(s)}{s}\, ds, \quad \text{for every } t \in [1, T].$$

3 Existence of solutions

Let us defining what we mean by a solution of problem (1)–(2).

Definition 3.1 A function $y \in C([1-r,T],E)$, is said to be a solution of (1)–(2) if y satisfies the equation $D^{\nu}y(t) = f(t, y_t, D^{\nu}y(t))$ on J, and the condition $y(t) = \varphi(t)$ on $[1-r,1]$.

To prove the existence of solutions to (1)–(2), we need the following auxiliary Lemma.

Lemma 3.2 *Let* $0 < \nu \leq 1$ *and let* $\sigma\colon J \to E$ *be a continuous function. The linear problem*

$$D^{\nu}y(t) = \sigma(t), \quad t \in J, \tag{4}$$

$$y(t) = \varphi(t), \ t \in [1-r,1], \tag{5}$$

has a unique solution which is given by:

$$y(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} \frac{\sigma(s)}{s} ds, & \textit{if } t \in J \\ \varphi(t), & \textit{if } t \in [1-r,1]. \end{cases} \tag{6}$$

First we list the following hypotheses:

(H1) The function $f\colon J \times C([-r,0],E) \times E \to E$ is continuous.

(H2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$||f(t,u,v) - f(t,\bar{u},\bar{v})|| \leq K||u-\bar{u}||_C + L||v-\bar{v}||$$

for any $u, \bar{u} \in C([-r,0],E)$, $v, \bar{v} \in E$ and $t \in J$.

We are now in a position to state and prove our existence result for the problem (1)–(2) based on concept of measures of noncompactness and *Darbo's* fixed point theorem.

Remark 3.3 ([7]) The condition $(H2)$ is equivalent to the inequality

$$\alpha\Big(f(t,B_1,B_2)\Big) \leq K\alpha(B_1) + L\alpha(B_2),$$

for any bounded sets $B_1 \subseteq C([-r,0],E), B_2 \subseteq E$, for each $t \in J$.

Theorem 3.4 *Assume (H1)–(H2) hold. If*

$$\frac{K\,(\log T)^{\nu}}{(1-L)\Gamma(\nu+1)} < 1 \tag{7}$$

then the IVP (1)–(2) *has at least one solution on* J.

Proof Transform the problem (1)-(2) into a fixed point problem. Consider the operator $N\colon C([1-r,T],E) \to C([1-r,T],E)$ defined by

$$Ny(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} g(s)\frac{ds}{s}, & t \in [1,T] \\ \varphi(t), & t \in [1-r,1], \end{cases} \tag{8}$$

where $g \in C(J,E)$ be such that

$$g(t) = f(t, y_t, g(t)).$$

Clearly, the fixed points of operator N are solutions of problem (1)–(2). We shall show that N satisfies the assumption of *Darbo's* fixed point Theorem. The proof will be given in several claims.

Claim 1: *N is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([1-r,T],E)$. If $t \in [1-r,1]$, then

$$\|N(u_n)(t) - N(u)(t)\| = 0.$$

For $t \in J$, we have

$$\|N(u_n)(t) - N(u)(t)\| \le \frac{1}{\Gamma(\nu)} \int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} \|g_n(s) - g(s)\| \frac{ds}{s}, \tag{9}$$

where $g_n, g \in C(J,E)$ such that

$$g_n(t) = f(t, u_{nt}, g_n(t)), \quad \text{and} \quad g(t) = f(t, u_t, g(t)).$$

By $(H2)$, we have

$$\begin{aligned} \|g_n(t) - g(t)\| &= \|f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))\| \\ &\le K\|u_{nt} - u_t\|_C + L\|g_n(t) - g(t)\|. \end{aligned}$$

Then

$$\|g_n(t) - g(t)\| \le \frac{K}{1-L}\|u_{nt} - u_t\|_C.$$

Since $u_n \to u$, then we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. And let $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \le \eta$ and $\|g(t)\| \le \eta$. Then, we have

$$\frac{1}{s}\left(\log \frac{t}{s}\right)^{\nu-1} \|g_n(s) - g(s)\| \le \frac{1}{s}\left(\log \frac{t}{s}\right)^{\nu-1} [\|g_n(s)\| + \|g(s)\|] \le \frac{2\eta}{s}\left(\log \frac{t}{s}\right)^{\nu-1}.$$

For each $t \in J$, the function $s \to \frac{2\eta}{s}\left(\log \frac{t}{s}\right)^{\nu-1}$ is integrable on $[1,t]$, then the Lebesgue Dominated Convergence Theorem and (9) imply that

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \; as \; n \to \infty,$$

and hence

$$\|N(u_n) - N(u)\|_{[1-r,T]} \to 0 \text{ as } n \to \infty.$$

Consequently, N is continuous.

Let the constant R such that:

$$R \geq \max\left\{\frac{f^*\,(\log T)^\nu}{(1-L)\Gamma(\nu+1) - K\,(\log T)^\nu}, \|\varphi\|_C\right\} \tag{10}$$

where $f^* = \sup_{t\in J}\|f(t,0,0)\|$.

Define

$$D_R = \{u \in C([1-r,T],E) : \|u\|_{[1-r,T]} \leq R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $C([1-r,T],E)$.

Claim 2: *$N(D_R) \subset D_R$.*

Let $u \in D_R$ we show that $Nu \in D_R$.

If $t \in [1-r,1]$ then $\|N(u)(t)\| \leq \|\varphi\|_C \leq R$. And if $t \in J$, we have

$$\|N(u)(t)\| \leq \frac{1}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1}\|g(s)\|\frac{ds}{s}. \tag{11}$$

By (H2) we have for each $t \in J$,

$$\begin{aligned}||g(t)|| &\leq \|f(t,u_t,g(t)) - f(t,0,0)\| + \|f(t,0,0)\| \leq K\|u_t\|_C + L\|g(t)\| + f^* \\ &\leq K\|u\|_{[1-r,T]} + L||g(t)|| + f^* \leq KR + L||g(t)|| + f^*.\end{aligned}$$

Then

$$||g(t)|| \leq \frac{f^* + KR}{1-L} := M.$$

Thus, (10) and (11) imply that

$$||Nu(t)|| \leq \frac{M}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1}\frac{ds}{s} \leq \frac{M(\log T)^\nu}{\Gamma(\nu+1)} \leq R$$

from which it follows that for each $t \in [-r,T]$, we have $\|Nu(t)\| \leq R$, which implies that $\|Nu\|_{[1-r,T]} \leq R$. Consequently, $N(D_R) \subset D_R$.

Claim 3: *$N(D_R)$ is bounded and equicontinuous.*

By Claim 2 we have $N(D_R) = \{N(u)\colon u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_{[1-r,T]} \leq R$ which means that $N(D_R)$ is bounded. Let $t_1, t_2 \in [1,T], t_1 < t_2$, and let $u \in D_R$. Then

$$\|N(u)(t_2) - N(u)(t_1)\| \leq$$

$$\leq \frac{1}{\Gamma(\nu)}\left\|\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\nu-1} - \left(\log\frac{t_1}{s}\right)^{\nu-1}\right]g(s)\frac{ds}{s} + \int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\nu-1}g(s)\frac{ds}{s}\right\|$$

$$\leq \frac{M}{\Gamma(\nu)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\nu-1} - \left(\log\frac{t_1}{s}\right)^{\nu-1}\right]\frac{ds}{s} + \frac{M}{\Gamma(\nu)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\nu-1}\frac{ds}{s}.$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 1$ and $t_1 \leq 1 \leq t_2$ is obvious.

Claim 4: *The operator $N\colon D_R \to D_R$ is a strict set contraction.*
Let $V \subset D_R$. If $t \in [1-r, 1]$, then

$$\alpha(N(V)(t)) = \alpha(N(y)(t), y \in V) = \alpha(\varphi(t), y \in V) = 0.$$

And if $t \in J$, we have

$$\alpha(N(V)(t)) = \alpha((Ny)(t), y \in V) \leq \frac{1}{\Gamma(\nu)}\left\{\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1} \alpha(g(s))\frac{ds}{s}, y \in V\right\}.$$

Then Remark 3.3 and Lemma 2.5 imply that, for each $s \in J$,

$$\begin{aligned}\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\})\\ &\leq K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}).\end{aligned}$$

Thus

$$\alpha\left(\{g(s), y \in V\}\right) \leq \frac{K}{1-L}\alpha\{y(s), y \in V\}.$$

Then

$$\begin{aligned}\alpha(N(V)(t)) &\leq \frac{K}{(1-L)\Gamma(\nu)}\left\{\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1} \{\alpha(y(s))\}\frac{ds}{s},\ y \in V\right\}\\ &\leq \frac{K\left(\log T\right)^{\nu}}{(1-L)\Gamma(\nu+1)}\alpha_c(V).\end{aligned}$$

Therefore

$$\alpha_c(NV) \leq \frac{K\left(\log T\right)^{\nu}}{(1-L)\Gamma(\nu+1)}\alpha_c(V).$$

So, by (7), the operator N is a set contraction. As a consequence of Theorem 2.6, we deduce that N has a fixed point which is solution to the problem (1)–(2). This completes the proof. □

Our next existence result for the problem (1)–(2) is based on concept of measures of noncompactness and *Mönch's* fixed point theorem.

Theorem 3.5 *Assume (H1)–(H4) and* (7) *hold. Then the IVP* (1)–(2) *has at least one solution.*

Proof Consider the operator N defined in (8). We shall show that N satisfies the assumption of *Mönch's* fixed point theorem. We know that $N\colon D_R \to D_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{\mathrm{conv}}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{\mathrm{conv}}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $[1-r, T]$. By Remark 3.3, Lemma 2.9 and the properties of the measure α we have for each $t \in J$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \leq \alpha(N(V)(t)) \leq \alpha\{(Ny)(t),\ y \in V\} \\ &\leq \frac{K}{(1-L)\Gamma(\nu)} \left\{ \int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} \{\alpha(y(s))\} \frac{ds}{s}, y \in V \right\} \\ &= \frac{K}{(1-L)\Gamma(\nu)} \int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} v(s)\, \frac{ds}{s}. \end{aligned}$$

Then

$$v(t) \leq \frac{K}{(1-L)\Gamma(\nu)} \int_1^t \left(\log \frac{t}{s}\right)^{\nu-1} v(s)\, \frac{ds}{s}.$$

Lemma 2.11 implies that $v(t) = 0$ for each $t \in J$.

For $t \in [1-r, 1]$ we have $v(t) = \alpha(\varphi(t)) = 0$, then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 2.8 we conclude that N has a fixed point $y \in D_R$. Hence N has a fixed point which is solution to the problem (1)–(2). This completes the proof. □

4 An example

Consider the following infinite system

$$D^{\frac{1}{2}} y_n(t) = \frac{1}{200}(t \sin(y_{n_t}) - y_{n_t} \cos(t)) + \frac{1}{100} \sin\left(D^{\frac{1}{2}} y_n(t)\right) + \frac{1}{2}, \quad \text{for each } t \in [1, e]. \tag{12}$$

$$y_n(t) = \varphi(t),\ t \in [1-r, 1],\ r > 0, \tag{13}$$

where $\varphi \in C([1-r, 1], E)$, and $\varphi(1) = 0$.

Set

$$E = l^1 = \left\{ y = (y_1, y_2, \ldots, y_n, \ldots), \sum_{n=1}^{\infty} |y_n| < \infty \right\},$$

and

$$f(t, u, v) = \frac{1}{200}(t \sin u - u \cos(t)) + \frac{1}{100} \sin v + \frac{1}{2}, \quad t \in [1, e],\ u \in C([-r, 0], E) \text{ and } v \in E.$$

Clearly, the function f is jointly continuous. E is a Banach space with the norm

$$||y|| = \sum_{n=1}^{\infty} |y_n|.$$

For any $u, \bar{u} \in C([-r,0],E)$, $v, \bar{v} \in E$ and $t \in [1,e]$:

$$\begin{aligned}
&\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \leq \\
&\quad \leq \frac{1}{200}|t|\|\sin u - \sin\bar{u}\| + \frac{1}{200}|\cos t|\|u-\bar{u}\| + \frac{1}{100}\|\sin v - \sin\bar{v}\| \\
&\quad \leq \frac{e}{200}\|u-\bar{u}\| + \frac{1}{200}\|u-\bar{u}\| + \frac{1}{100}\|v-\bar{v}\| \\
&\quad = \frac{e+1}{200}\|u-\bar{u}\| + \frac{1}{100}\|v-\bar{v}\|.
\end{aligned}$$

Hence condition (H2) is satisfied with

$$K = \frac{e+1}{200} \quad \text{and} \quad L = \frac{1}{100}.$$

And the conditions

$$\frac{K(\log T)^{\nu}}{(1-L)\Gamma(\nu+1)} = \frac{\frac{e+1}{200}}{(1-\frac{1}{100})\Gamma(\frac{3}{2})} = \frac{e+1}{99\sqrt{\pi}} < 1,$$

are satisfied with

$$T = e \quad \text{and} \quad \nu = \frac{1}{2}.$$

It follows from Theorem 3.4 that the problem (12)–(13) has a at least one solution on J.

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