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Hypercomplex Algebras and Geometry of Spaces with Fundamental Form of an Arbitrary Order

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Abstract

The article is devoted to a generalization of Clifford and Grassmann algebras for the case of vector spaces over the field of complex numbers. The geometric interpretation of such generalizations are presented. Multieuclidean geometry is considered as well as the importance of it in physics.

Key words: Hypercomplex algebras, geometry of spaces with fundamental form, Clifford algebras.

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It is well known that Clifford and Grassmann algebras play an important role in different branches of geometry (see [1]). In this article, using a non-quadratic fundamental form we construct certain generalizations of such algebras over complex numbers and bring the geometric interpretation of them.

Let us consider an n-dimensional vector space $\boldsymbol{V}$ over complex numbers $\mathbb{C}$. Let

$$\boldsymbol{Q}(\boldsymbol{x}) = x_1^m + x_2^m + \cdots + x_k^m$$

be a homogenous form, $0 \le k \le n$, $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ be some basis of $\boldsymbol{V}$and $\boldsymbol{x} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_n$. Using this form we construct the associative unitary algebra $\boldsymbol{B}(\boldsymbol{V}, \boldsymbol{Q})$, the basis of which is formed by monomials

$$\boldsymbol{e}_{a_1 a_2 \dots a_n} \equiv \boldsymbol{e}_1^{a_1} \cdot \boldsymbol{e}_2^{a_2} \cdot \ \dots \ \cdot \boldsymbol{e}_n^{a_n}, \quad 0 \le a_q \le m-1, \quad \boldsymbol{e}_1^0 \cdot \boldsymbol{e}_2^0 \cdot \ \dots \ \cdot \boldsymbol{e}_n^0 \equiv 1,$$

multiplication is given by commutation relations

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \alpha_m \boldsymbol{e}_b \cdot \boldsymbol{e}_a, \quad a > b, \qquad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \overline{\alpha}_m \boldsymbol{e}_b \cdot \boldsymbol{e}_a, \quad a < b, \tag{1}$$

where α_m is m-th primitive root of unity—for example

$$\alpha_m = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m},$$

and by identities

$$\boldsymbol{e}_a^m = \boldsymbol{Q}(\boldsymbol{e}_a) \text{ i. e. } \boldsymbol{e}_1^m = \cdots = \boldsymbol{e}_k^m = 1,\ \boldsymbol{e}_{k+1}^m = \cdots = \boldsymbol{e}_n^m = 0. \tag{2}$$

The dimension of the algebra $\boldsymbol{B}(\boldsymbol{V}, \boldsymbol{Q})$ is equal to m^n. The algebra constructed by this way will be called a generalized exterior algebra with fundamental form $\boldsymbol{Q}$ (see [2]). It is evident that for $k = n$, $m = 2$ we get a Clifford algebra. For arbitrary natural numbers m, the generalized exterior algebra will be called an *elementary algebra* of order m and it will be denoted by $\boldsymbol{B}_n^m$. For the case $k = 0$, $m = 2$ we obtain a Grassmann algebra; for arbitrary natural numbers it will be called a *radical algebra* of order m and it will be denoted by $\boldsymbol{X}_n^m$ (see [3]). The following theorem expresses the basic property of generalized exterior algebras.

Theorem 1 *For every vector*

$$\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \cdots + x_n \boldsymbol{e}_n \in \boldsymbol{V} \subset \boldsymbol{B}(\boldsymbol{V}, \boldsymbol{Q})$$

the following identity holds

$$\boldsymbol{x}^m \equiv (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \cdots + x_n \boldsymbol{e}_n)^m = x_1^m \boldsymbol{e}_1^m + x_2^m \boldsymbol{e}_2^m + \cdots + x_n^m \boldsymbol{e}_n^m \equiv \boldsymbol{Q}(\boldsymbol{x}). \tag{3}$$

Proof We prove this theorem by mathematical induction for dimension of vector space $\boldsymbol{V} \subset \boldsymbol{B}(\boldsymbol{V}, \boldsymbol{Q})$.

Let $n = 2$, $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \cdots + x_n \boldsymbol{e}_n \in \boldsymbol{V}_2 \subset \boldsymbol{B}(\boldsymbol{V}, \boldsymbol{Q})$. Then we may write:

$$\begin{aligned}(x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^m = x_1^m \boldsymbol{e}_1^m &+ p_1(\alpha_m) x^{m-1} x_2 \boldsymbol{e}_1^{m-1} \cdot \boldsymbol{e}_2 \\ &+ \cdots + p_r(\alpha_m) x_1^{m-r} x_2 \boldsymbol{e}_1^{m-r} \cdot \boldsymbol{e}_2^r \\ &+ \cdots + p_{m-1}(\alpha_m) x_1 x_2^{m-1} \boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} + x_2 \boldsymbol{e}_2^m,\end{aligned}$$

where $p_r(\alpha_m)$ is some polynomial of α_m. Let us prove $p_r(\alpha_m) = 0$.

Using

$$(x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (y_1 \boldsymbol{e}_1 + y_2 \boldsymbol{e}_2) = (\alpha_m y_1 \boldsymbol{e}_1 + y_2 \boldsymbol{e}_2) \cdot (\overline{\alpha}_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2),$$

we may write

$$(x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^m = (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^{m-2} =$$

$$= (\alpha_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (\overline{\alpha}_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^{m-2} =$$

$$= (\alpha_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (\overline{\alpha}_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^{m-3} =$$

$$= (\alpha_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^2 \cdot (\overline{\alpha}_m^2 x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) \cdot (x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^{m-3} = \cdots =$$

$$= (\alpha_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^{m-1} \cdot (\overline{\alpha}_m^{m-1} x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2) = (\alpha_m x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2)^m,$$

which means

$$(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2)^m = (\alpha_m x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2)^m,$$

and

$$\sum_{r=0}^{m} x_1^{m-r}x_2^r p_r(\alpha_m)\boldsymbol{e}_1^{m-r}\cdot\boldsymbol{e}_2^r = \sum_{r=0}^{m} x_1^{m-r}x_2^r\alpha_m^r p_r(\alpha_m)\boldsymbol{e}_1^{m-r}\cdot\boldsymbol{e}_2^r.$$

Therefore

$$p_r(\alpha_m) = \alpha_m^r p_r(\alpha_m) \quad \text{or} \quad (1-\alpha_m^r)p_r(\alpha_m) = 0,$$

which implies that $p_r(\alpha_m) = 0$ because of $1-\alpha_m^r \neq 0$. We have got

$$(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2)^m = x_1^m\boldsymbol{e}_1^m + x_2^m\boldsymbol{e}_2^m \equiv \boldsymbol{Q}(\boldsymbol{x}).$$

Let us suppose that the theorem holds for $n = l$:

$$(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \dots + x_l\boldsymbol{e}_l)^m \equiv \boldsymbol{Q}(\boldsymbol{x}),$$

and prove the theorem for $n = l+1$. Since elements $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \dots + x_l\boldsymbol{e}_l$ and $\boldsymbol{e}_{l+1}$ fulfill commutation relations (1) in the form

$$\boldsymbol{e}_{l+1}\cdot\boldsymbol{x} = \alpha_m\boldsymbol{x}\cdot\boldsymbol{e}_{l+1} \quad \text{and} \quad \boldsymbol{x}\cdot\boldsymbol{e}_{l+1} = \overline{\alpha}_m\cdot\boldsymbol{x},$$

we obtain

$$\begin{aligned}(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \dots + x_l\boldsymbol{e}_l + x_{l+1}\boldsymbol{e}_{l+1})^m &\equiv (\boldsymbol{x} + x_{l+1}\boldsymbol{e}_{l+1})^m \\ &= \boldsymbol{x}^m + x_{l+1}^m\boldsymbol{e}_{l+1}^m \equiv \boldsymbol{Q}(\boldsymbol{x} + x_{l+1}\boldsymbol{e}_{l+1}).\end{aligned}$$

The theorem has been proved. □

Now, let us investigate the algebra $\boldsymbol{B}_2^m$ more detail. According to [3], this algebra may be represented as a *noncommutative composition* of two cyclic algebras $\boldsymbol{C}_m(\boldsymbol{e}_1)$ and $\boldsymbol{C}_m(\boldsymbol{e}_2)$, for example. Thus any element $\boldsymbol{x} \in \boldsymbol{B}_2^m$ may be expressed by

$$\boldsymbol{x} = \boldsymbol{y}_0 + \boldsymbol{y}_1\boldsymbol{e}_2 + \boldsymbol{y}_2\boldsymbol{e}_2^2 + \dots + \boldsymbol{y}_{m-1}\boldsymbol{e}_2^{m-1}, \tag{4}$$

where $\boldsymbol{y}_l = z_{l0} + z_{l1}\boldsymbol{e}_1 + \dots + z_{lm-1}\boldsymbol{e}_1^{m-1}$ are *cyclic coordinates* of the element $\boldsymbol{x} \in \boldsymbol{B}_2^m$ and $z_{lr} \in \mathbb{C}$. In this case, using commutation relations (1) we may define the multiplication of elements of algebra $\boldsymbol{B}_2^m$ which are written in the form (4) by the following commutation identities

$$\boldsymbol{e}_m^r\cdot\boldsymbol{y}_l = \boldsymbol{y}_l(\alpha_m^r)\cdot\boldsymbol{e}_2, \tag{5}$$

where $\boldsymbol{y}_l(\alpha_m) = z_{l0} + \alpha_m^r z_{l1}\boldsymbol{e}_1 + \alpha_m^{2r}z_{l2}\boldsymbol{e}_1^2 + \dots + \alpha_m^{(m-1)r}z_{lm-1}\boldsymbol{e}_1^{m-1}$. Noncommutative composition of algebras $\boldsymbol{C}_m(\boldsymbol{e}_1)$ and $\boldsymbol{C}_m(\boldsymbol{e}_2)$ will be denoted by $\boldsymbol{C}_m(\boldsymbol{e}_1)\star\boldsymbol{C}_m(\boldsymbol{e}_2)$, i. e.

$$\boldsymbol{B}_2^m \equiv \boldsymbol{C}_m(\boldsymbol{e}_1)\star\boldsymbol{C}_m(\boldsymbol{e}_2). \tag{6}$$

Now let us give some definitions.

Definition 1

a) *Let* $\boldsymbol{x} = x_0 + x_1\boldsymbol{e} + x_2\boldsymbol{e}^2 + \dots + x_{m-1}\boldsymbol{e}^{m-1}$ *be an arbitrary element of the cyclic algebra* $\boldsymbol{C}_m(\boldsymbol{e})$ *with a generator* $\boldsymbol{e}$ *fulfilling identity* $\boldsymbol{e}^m = 1$*. Then the element*

$$\boldsymbol{x}(\alpha_m) = x_0 + \alpha_m x_1\boldsymbol{e} + \alpha_m^2 x_2\boldsymbol{e}^2 + \dots + \alpha_m^{m-1}x_{m-1}\boldsymbol{e}^{m-1}$$

will be called a resolvent of the element $\boldsymbol{x} \in \boldsymbol{C}_m(\boldsymbol{e})$*. An operator* $\hat{\alpha}_m : \boldsymbol{C}_m(\boldsymbol{e}) \mapsto \boldsymbol{C}_m(\boldsymbol{e})$ *such that* $\hat{\alpha}_m(\boldsymbol{x}) = \boldsymbol{x}(\alpha_m)$ *will be called a* resolvent operator*. In this case, the element*

$$\hat{\alpha}_m^r(\boldsymbol{x}) = \boldsymbol{x}(\alpha_m^r) = x_0 + \alpha_m^r x_1\boldsymbol{e} + \alpha_m^{2r}x_2\boldsymbol{e}^2 + \dots + \alpha_m^{(m-1)r}x_{(m-1)r}\boldsymbol{e}^{m-1}$$

will be called a resolvent of the order r*.*

b) *The value*

$$\Delta(\boldsymbol{a}) = \begin{vmatrix} a_0 & a_{m-1} & a_{m-2} & \dots & a_1 \\ a_1 & a_0 & a_{m-1} & \dots & a_2 \\ a_2 & a_1 & a_0 & \dots & a_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m-1} & a_{m-2} & a_{m-3} & \dots & a_0 \end{vmatrix}$$

will be called a determinant of the element $\boldsymbol{a} = a_0 + a_1\boldsymbol{e} + a_2\boldsymbol{e}^2 + \dots + a_{m-1}\boldsymbol{e}^{m-1} \in \boldsymbol{C}_m(\boldsymbol{e})$*.*

c) *The value* $\mathrm{sp}(\boldsymbol{a}) = a_0 + a_1 + a_2 + \dots + a_{m-1}$ *will be called a* trace *of the element* $\boldsymbol{a} \in \boldsymbol{C}_m(\boldsymbol{e})$*.*

d) *The value*

$$\Delta_R(\boldsymbol{b}) = \begin{vmatrix} b_0 & b_{m-1}(\alpha_m) & b_{m-2}(\alpha_m^2) & \dots & b_1(\alpha_m^{m-1}) \\ b_1 & b_0(\alpha_m) & b_{m-1}(\alpha_m^2) & \dots & b_2(\alpha_m^{m-1}) \\ b_2 & b_1(\alpha_m) & b_0(\alpha_m^2) & \dots & b_3(\alpha_m^{m-1}) \\ \dots & \dots & \dots & \dots & \dots \\ b_{m-1} & b_{m-2}(\alpha_m) & b_{m-3}(\alpha_m^2) & \dots & b_0(\alpha_m^{m-1}) \end{vmatrix}$$

will be called a right cyclic determinant *of the element*

$$\boldsymbol{b} = \boldsymbol{b}_0 + \boldsymbol{b}_1\boldsymbol{e}_2 + \boldsymbol{b}_2\boldsymbol{e}_2^2 + \dots + \boldsymbol{b}_{m-1}\boldsymbol{e}_2^{m-1} \in \boldsymbol{B}_2^m \equiv \boldsymbol{C}_m(\boldsymbol{e}_1) \star \boldsymbol{C}_m(\boldsymbol{e}_2),$$

where $\boldsymbol{b}_l \in \boldsymbol{C}_m(\boldsymbol{e}_1)$*.*

By direct calculations it may be easy proved that the notions defined above have properties that are presented in the following theorems (see [3]).

Theorem 2

a) *The resolvent operator is an multiinvolutory endomorphism of a cyclic algebra. It means* $\hat{\alpha}_m(\boldsymbol{x}+\boldsymbol{y}) = \hat{\alpha}_m(\boldsymbol{x})+\hat{\alpha}_m(\boldsymbol{y})$, $\hat{\alpha}_m(\boldsymbol{x}\cdot\boldsymbol{y}) = \hat{\alpha}_m(\boldsymbol{x})\cdot\hat{\alpha}_m(\boldsymbol{y})$ *and* $\hat{\alpha}_m^m(\boldsymbol{x}) = \mathrm{id}$*.*

b) The following identities hold

$$\Delta(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{a}(\alpha_m) \cdot \boldsymbol{a}(\alpha_m^2) \cdot \ \ldots \ \cdot \boldsymbol{a}(\alpha_m^{m-1})$$
$$= \mathrm{sp}\boldsymbol{a}\ \mathrm{sp}\boldsymbol{a}(\alpha_m)\ \mathrm{sp}\boldsymbol{a}(\alpha_m^2) \ldots \mathrm{sp}\boldsymbol{a}(\alpha_m^{m-1}). \quad (7)$$

c) The determinant of an element $\boldsymbol{a} \in \boldsymbol{C}_m(\boldsymbol{e})$ represents the determinant of a system of linear scalar equations which is equivalent to the algebraic equation $\boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{c}$.

d) In cyclic algebras, the determinant is a multiplicative functions; it means that for every $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{C}_m(\boldsymbol{e})$ it holds

$$\Delta(\boldsymbol{a} \cdot \boldsymbol{b}) = \Delta(\boldsymbol{a})\Delta(\boldsymbol{b}). \quad (8)$$

e) The right cyclic determinant of an element $\boldsymbol{a} \in \boldsymbol{B}_n^m$ represents the determinant of a system of linear equations which is equivalent to the linear algebraic equation $\boldsymbol{a} \cdot \boldsymbol{x} = \boldsymbol{c}$ that is written in cyclic coordinates.

f) In algebras $\boldsymbol{B}_n^m$, the right determinant is a multiplicative function; it means that for every $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{B}_n^m$ it holds

$$\Delta_R(\boldsymbol{a} \cdot \boldsymbol{b}) = \Delta_R(\boldsymbol{a})\Delta_R(\boldsymbol{b}). \quad (9)$$

g) Let $\boldsymbol{C}(\boldsymbol{e}_1) \star \boldsymbol{C}(\boldsymbol{e}_2) \equiv \boldsymbol{B}_2^m$. If $\boldsymbol{a} \in \boldsymbol{C}(\boldsymbol{e}_1)$ then $\Delta_R(\boldsymbol{a}) = \Delta(\boldsymbol{a})$ and if $\boldsymbol{b} \in \boldsymbol{C}_(\boldsymbol{e}_2)$ then $\Delta_R(\boldsymbol{b}) = \Delta(\boldsymbol{b})$, where $\Delta(\boldsymbol{a}), \Delta(\boldsymbol{b})$ are determinants in cyclic algebras $\boldsymbol{C}(\boldsymbol{e}_1), \boldsymbol{C}(\boldsymbol{e}_2)$.

Further, let us study an analytic continuation of entire functions of a complex variable into a cyclic algebra $\boldsymbol{C}_m(\boldsymbol{e})$; we obtain it by putting variable $\boldsymbol{x} \in \boldsymbol{C}_m(\boldsymbol{e})$ into a power series of the given function of complex variable. Especially, for an exponential function of variable $t\boldsymbol{e} \in \boldsymbol{C}_m(\boldsymbol{e})$, $t \in \mathbb{C}$ we get:

$$\exp(t\boldsymbol{e}) = E_0^m(t) + \boldsymbol{e}E_1^m(t) + \boldsymbol{e}^2E_2^m(t) + \cdots + \boldsymbol{e}^{m-1}E_{m-1}^m(t), \quad (10)$$

where functions $E_r^m(t)$ are defined by the following series

$$E_r^m(t) = \sum_{q=0}^{\infty} \frac{x^{mq+r}}{(mq+r)!}.$$

Analogically, we obtain the expression for exponential function of variable $t\boldsymbol{e}^p$. For a *full cyclic variable* $\boldsymbol{x} = x_0 + x_1\boldsymbol{e} + x_2\boldsymbol{e}^2 + \cdots + x_{m-1}\boldsymbol{e}^{m-1}$ the following identity holds

$$\exp \boldsymbol{x} = \prod_{r=0}^{m-1} \exp(x_r\boldsymbol{e}^r). \quad (11)$$

For exponential functions of cyclic variables we have the following theorem.

Theorem 3 *For every element* $\boldsymbol{x} = x_0 + x_1\boldsymbol{e} + x_2\boldsymbol{e}^2 + \cdots + x_{m-1}\boldsymbol{e}^{m-1}$ *it holds*

$$\Delta(\exp \boldsymbol{x}) = 1. \tag{12}$$

Proof For $\boldsymbol{x} = x_r\boldsymbol{e}^r$ the identity (12) follows from the relation (7) and from the fact $1+\alpha_m+\alpha_m^2+\cdots+\alpha_m^{m-1} = 0$. Thus for $\boldsymbol{x} = x_0+x_1\boldsymbol{e}+x_2\boldsymbol{e}^2+\cdots+x_{m-1}\boldsymbol{e}^{m-1}$ the identity (12) follows from the relation (11), immediately. □

Using Theorems 2d and 3 we can for cyclic algebras introduce a geometric structure where the fundamental form is represented by the determinant of a variable element and motions are given by linear algebraic functions

$$\boldsymbol{x}' = (\exp \xi) \cdot \boldsymbol{x} \equiv (\exp(\xi_1\boldsymbol{e} + \xi_2\boldsymbol{e}^2 + \cdots + \xi_{m-1}\boldsymbol{e}^{m-1})) \cdot \boldsymbol{x}. \tag{13}$$

The correctness of such definition of motions in the linear space of this cyclic algebra follows from identities (8) and (12) because the determinant of an variable element of cyclic algebra is invariant with respect to any transformation which is given by the relation (13).

Any geometry defined on the linear space of a cyclic algebra $\boldsymbol{C}_m(\boldsymbol{e})$ by the fundamental form $\Delta(\boldsymbol{x})$ and group of motions which are given by functions (13) will be called a *cyclic geometry of the order m*. In such geometry, we define a *cyclic norm of the vector* represented by $\boldsymbol{x} \in \boldsymbol{C}_m(\boldsymbol{e})$ by the following relation:

$$|\boldsymbol{x}|^2 = \Delta(\boldsymbol{x})\overline{\Delta(\boldsymbol{x})},$$

where $\overline{\Delta(\boldsymbol{x})}$ is a form complex conjugate to $\Delta(\boldsymbol{x})$. It is clear to see that such norm is invariant with respect to any motion. In a cyclic geometry it is possible to define a measured angle of two vectors represented by elements $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{C}_m(\boldsymbol{e})$ bounded up by some motion; if $\boldsymbol{y} = (\exp\xi) \cdot \boldsymbol{x}$, then ξ can be considered as a *cyclic angle* between given vectors. Thus, motions given by linear functions in the form (13) get the meaning of *cyclic rotations*.

Structures of cyclic geometries defined on algebras $\boldsymbol{C}_m(\boldsymbol{e}_1)$ and $\boldsymbol{C}_m(\boldsymbol{e}_2)$ generate a geometris structure on the algebra $\boldsymbol{B}_2^m \equiv \boldsymbol{C}_m(\boldsymbol{e}_1) \star \boldsymbol{C}_m(\boldsymbol{e}_2)$ with the fundamental form $\Delta_R(\boldsymbol{x})$ and motions which are defined by linear algebraic functions of the following form

$$\boldsymbol{x}' = (\exp \xi) \cdot \boldsymbol{x} \cdot (\exp \eta), \tag{14}$$

where ξ and η are arbitrary cyclic variables with zero real part. This choice of transformations in the meaning of motions is correct due to Theorems 2g and 3 since for determinant $\Delta_R(\boldsymbol{x})$ it holds $\Delta_R(\boldsymbol{x}') = \Delta_R(\boldsymbol{x})$.

Now, let us show that the geometric structure of the algebra $\boldsymbol{B}_2^m$ as defined above may be extended to the case of arbitrary algebra $\boldsymbol{B}_n^m$. It is important to see that algebra $\boldsymbol{B}_3^m$ is a *commutative composition* of an algebra $\boldsymbol{B}_2^m(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and a cyclic algebra $\boldsymbol{C}_m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_3)$ because the element $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_3$ commutes with basis elements $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of the algebra $\boldsymbol{B}_2^m(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Commutative composition of algebras $\boldsymbol{B}_2^m(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and $\boldsymbol{C}_m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_3)$ will be denoted by

$$\boldsymbol{B}_3^m \equiv \boldsymbol{B}_2^m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2) \circ \boldsymbol{C}_m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_3).$$

By an analogical way, the algebra $\boldsymbol{B}_4^m$ is a commutative composition

$$\boldsymbol{B}_4^m \equiv \boldsymbol{B}_2^m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2) \circ \boldsymbol{B}_2^m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_4),$$

etc. By this way, we will obtain representations of arbitrary algebras $\boldsymbol{B}_n^m$ in the following form of commutative compositions:

$$\boldsymbol{B}_n^m = \begin{cases} \boldsymbol{B}_2^m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2) \circ \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_3, \tilde{\boldsymbol{e}}_4) \circ \cdots \circ \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_{2r-1}, \tilde{\boldsymbol{e}}_{2r}), & \text{if } n = 2r \\ \boldsymbol{B}_2^m(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2) \circ \cdots \circ \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_{2r-1}, \tilde{\boldsymbol{e}}_{2r}) \circ \boldsymbol{C}_m(\tilde{\boldsymbol{e}}_{2r+1}), & \text{if } n = 2r+1 \end{cases}, \tag{15}$$

where $\tilde{\boldsymbol{e}}_3 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_3$, $\tilde{\boldsymbol{e}}_4 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2^{m-1} \cdot \boldsymbol{e}_4$, $\tilde{\boldsymbol{e}}_5 = \tilde{\boldsymbol{e}}_3 \cdot \tilde{\boldsymbol{e}}_4^{m-1} \cdot \boldsymbol{e}_5$, $\tilde{\boldsymbol{e}}_6 = \tilde{\boldsymbol{e}}_3 \cdot \tilde{\boldsymbol{e}}_4^{m-1} \cdot \boldsymbol{e}_6$, etc. The relation (15) generalizes a well known representation of Clifford algebras in the form of commutative compositions of algebras of complex quaternions and algebras of dual complex numbers.

Now, let us investigate a linear space

$$\boldsymbol{E} = \boldsymbol{B}_2^m(\boldsymbol{e}_1, \boldsymbol{e}_2) \oplus \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_3, \tilde{\boldsymbol{e}}_4) \oplus \cdots \oplus \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_{2r-1}, \tilde{\boldsymbol{e}}_{2r})$$

on which a form of order m is defined by

$$\boldsymbol{g}_m(\boldsymbol{x}) = \Delta_R(\boldsymbol{x}_1) + \Delta_R(\boldsymbol{x}_2) + \cdots + \Delta_R(\boldsymbol{x}_r), \tag{16}$$

where $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_r$, $\boldsymbol{x}_l \in \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_{2l-1}, \tilde{\boldsymbol{e}}_{2l})$. This form is invariant with respect to the transformations of the following form

$$\boldsymbol{x}' = (\exp \xi_l) \cdot \boldsymbol{x} \cdot (\exp(-\xi_l)), \tag{17}$$

where $\xi_l \in \boldsymbol{B}_2^m(\tilde{\boldsymbol{e}}_{2l-1}, \tilde{\boldsymbol{e}}_{2l})$ are cyclis variables with zero scalar part since

$$\begin{aligned} \boldsymbol{g}_m(\boldsymbol{x}') &= \boldsymbol{g}_m(\boldsymbol{x}_1 + \cdots + (\exp \xi_l) \cdot \boldsymbol{x}_l \cdot (\exp(-\xi_l)) + \cdots + \boldsymbol{x}_r) \\ &= \Delta_R(\boldsymbol{x}_1) + \cdots + \Delta_R(\exp \xi_l) \Delta_R(\boldsymbol{x}_l) \cdot \Delta_R(\exp(-\xi_l)) + \cdots + \Delta_R(\boldsymbol{x}_r) = \boldsymbol{g}_m(\boldsymbol{x}). \end{aligned}$$

The group of transformations, which are generated by linear algebraic function of the type (17), defines a geometric structure on a linear space $\boldsymbol{E}$ with fundamental form (16). For $m = 2$ this group is identical to the group of spinors of a complex Euclidean space (see [5]). Therefore we can call the geometry of a space $\boldsymbol{E}$ with the fundamental form (16) a *multieuclidean geometry* of the order m.

A multieuclidean geometry have been here introduced by an analogy with the geometry of a complex Euclidean space, the geometry of which is given by a group of spinors. In this sense, the multieuclidean geometry represents some abstract geometric construction. However, such geometry may be for $m \geq 3$ used for a description of spaces of symmetry of hadrons, the basis elements of which may be identified with different generations of quarks, because the system of hadrons admits the presence of several quarks of one generation (see [4]).

Let us conclude, that geometric structures with the fundamental form of an arbitrary order m may be (locally) defined also on smooth manifolds M in the case when on a tangent bundle there is given a linear algebra structure the restriction of which on any tangent space $\boldsymbol{T}_X$, $X \in \boldsymbol{M}$ gives the algebra $\boldsymbol{B}(\boldsymbol{T}_X, \boldsymbol{Q})$.

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