Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **55**, 1 (2016) 39–45

A Note on Pseudo-Kleene Algebras

Ivan CHAJDA

Department of Algebra and Geometry, Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: ivan.chajda@upol.cz

(Received December 10, 2015)

Abstract

We introduce the concept of a pseudo-Kleene algebra which is a non-distributive modification of a Kleene algebra introduced by J. A. Kalman [4]. Basic properties of pseudo-Kleene algebras are studied. For pseudo-Kleene algebras with a fix-point there are determined subdirectly irreducible members.

Key words: Antitone involution, De Morgan laws, Kleene algebra, distributive lattice, pseudo-Kleene algebra, subdirectly irreducible.

2010 Mathematics Subject Classification: 06B05, 06D30

Lattices with an antitone involution, i.e. satisfying De Morgan laws, were introduced and treated already by G. Birkhoff [1]. He introduced also the concept of an orthocomplemented lattice. The concept of a Kleene algebra as a particular case of a lattice with an antitone involution was introduced by J. A. Kalman under the name *normal i-lattice* in [4]. Several important results on these algebras can be found in the paper by R. Cignoli [2]. There exists also another notion of Kleene algebra treated, e.g., in [3], which is an abstract analogue of certain algebras of binary relations, and therefore, it differs from algebras investigated here. Our aim is to generalize the concept of Kleene algebra also for lattices which need not be distributive and to show that such algebras, called *pseudo-Kleene algebras*, still have several important properties and a nice structure.

By a *lattice with an antitone involution* is meant an algebra $\mathscr{A} = (A; \vee, \wedge, ')$ such that $(A; \vee, \wedge)$ is a lattice and $'$ is a mapping of A into itself satisfying the conditions

$$(x')' = x \qquad \text{and} \qquad x \le y \Rightarrow y' \le x'.$$

Instead of $(x')'$ we will write only x''. The aforementioned conditions can be expressed equivalently by the De Morgan laws

$$(x \vee y)' = x' \wedge y', \quad (x \wedge y)' = x' \vee y' \quad \text{and} \quad x'' = x. \tag{1}$$

A distributive lattice with an antitone involution is called a *Kleene algebra* if it, moreover, satisfies the identity

$$x \wedge x' \leq y \vee y'. \tag{2}$$

A lattice with an antitone involution which is bounded (0 is the least and 1 the greatest element) is called an *orthocomplemented lattice* if it satisfies

$$x \wedge x' = 0 \qquad \text{and} \qquad x \vee x' = 1.$$

Of course, every orthocomplemented lattice satisfies the identity (2). However, an example of a Kleene algebra which is not orthocomplemented is visualized in Fig. 1.

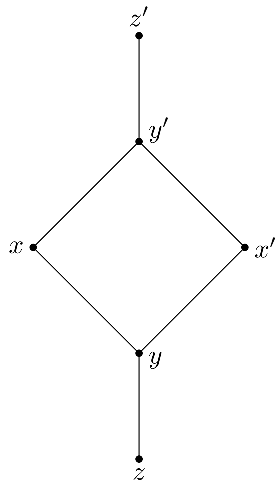

Fig. 1

Since every Kleene algebra is distributive, it satisfies the following identity

$$x \wedge (x' \vee y) = (x \wedge x') \vee (x \wedge y), \tag{3}$$

which is equivalent to the identity

$$x \vee (x' \wedge y) = (x \vee x') \wedge (x \vee y).$$

We can easily show that (3) does not imply that a given lattice with antitone involution is distributive, see the following

Example 1 Consider the lattice $\mathscr{L}$ with an antitone involution depicted in Fig. 2. Then clearly $\mathscr{L}$ satisfies the identities (1) and (3), but it is not distributive. Hence, it motivates us to introduce the following concept.

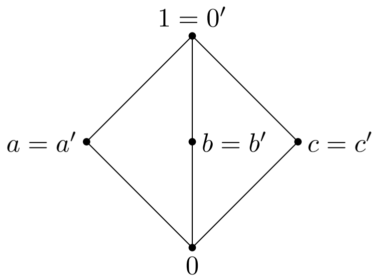

Fig. 2

Definition 1 *By a pseudo-Kleene algebra is meant a lattice with an antitone involution satisfying* (2) *and* (3)*.*

To justify our concept, we give an example.

Example 2 The lattice with an antitone involution depicted in Fig. 3 is a pseudo-Kleene algebra which is not a Kleene algebra.

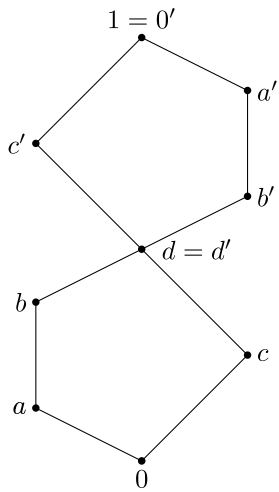

Fig. 3

Moreover, we can prove the following result.

Theorem 1 *For a lattice with an antitone involution, the identities* (2) *and* (3) *are independent.*

Proof The lattice with antitone involution in Fig. 2 satisfies (1) and (3) but not (2) because $a \wedge a' = a \not\leq b = b \vee b'$. The lattice with antitone involution visualized in Fig. 4 satisfies (1) and (2) but not (3) because

$$x \vee (x' \wedge y) = x \vee 0 = x \neq y = 1 \wedge y = (x \vee x') \wedge (x \vee y).$$

□

Let $\mathscr{L} = (L; \vee, \wedge, ')$ be a lattice with an antitone involution. An element $d \in L$ is called a *fix-point* if $d' = d$. The pseudo-Kleene algebra in Fig. 3 has a fix-point d. It is easy to prove the following result.

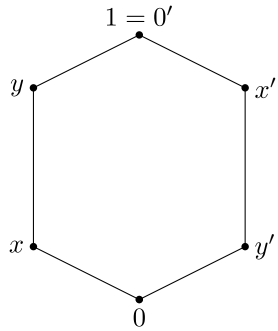

Fig. 4

Lemma 1 *A pseudo-Kleene algebra has at most one fix-point.*

Proof Assume b, c are fix-points of a pseudo-Kleene algebra. Using the identity (2) we obtain

$$b = b \wedge b = b \wedge b' \leq c \vee c' = c \vee c = c$$

and

$$c = c \wedge c = c \wedge c' \leq b \vee b' = b \vee b = b$$

proving $b = c$. □

Let us introduce the following construction. Consider a lattice $\mathscr{L} = (L; \vee, \wedge, 1)$ with the greatest element 1. Take $L^d = \{x^* \mid x \in L\}$ and define an inverse order on L^d, i.e. $x^* \leq y^*$ if and only if $y \leq x$ in $\mathscr{L}$. Then $\mathscr{L}^d = (L^d; \vee, \wedge, 1^*)$ is a lattice with the least element 1^*, the so-called *dual* of $\mathscr{L}$. Let $P = L \cup L^d$, where 1 is identified with 1^*. In P we consider the same order as in $\mathscr{L}$ for elements from L and the same order as in $\mathscr{L}^d$ for elements from L^d. Moreover, $x \leq y$ for each $x \in L$ and each $y \in L^d$. Hence, $(P; \vee, \wedge)$ is a lattice again. In P we introduce the involution $'$ as follows

$$x \longmapsto x^* \quad \text{and} \quad x^* \longmapsto x$$

for each $x \in L$. Then $\mathscr{P} = (P; \vee, \wedge, ')$ will be called the *composition of* $\mathscr{L}$ *and* $\mathscr{L}^d$.

We are going to show that every lattice with greatest element can be embedded into pseudo-Kleene algebra.

Theorem 2 *Let $\mathscr{L} = (L; \vee, \wedge, 1)$ be a lattice with the greatest element. Then there exists a pseudo-Kleene algebra with a fix-point having a sublattice isomorphic to $(L; \vee, \wedge)$.*

Proof Let $\mathscr{L} = (L; \vee, \wedge, 1)$ be a lattice with the greatest element 1. Consider its dual $\mathscr{L}^d$ and take $\mathscr{P}$ to be the composition of $\mathscr{L}$ and $\mathscr{L}^d$. Then it is evident that 1 is the fix-point of the antitone involution $'$ on $\mathscr{P}$. It is clear that $\mathscr{P}$ satisfies (1) and (2). All we need is to prove the identity (3).

(a) If $x, y \in L$ then clearly $x' \wedge y \in L$ and $x' \wedge y = y$ thus $x \vee (x' \wedge y) = x \vee y$. Since $x \vee x' \in L^d$, we have $x \vee x' \geq x \vee y$ and hence $(x \vee x') \wedge (x \vee y) = x \vee y = x \vee (x' \wedge y)$.

(b) Assume $x \in L$ and $y \in L^d$. Then $x' \in L^d$ thus also $x' \wedge y \in L^d$, i.e. $x \leq x' \wedge y$ giving $x \vee (x' \wedge y) = x' \wedge y$. However, $x \vee x' = x'$ and $x \vee y = y$ whence $(x \vee x') \wedge (x \vee y) = x' \wedge y = x \vee (x' \wedge y)$.

The proof of the case $x \in L^d,\ y \in L$ and of the case $x, y \in L^d$ can be done by dualization of the foregoing cases (b) and (a), respectively. □

It is a natural question how general the construction of composition of $\mathscr{L}$ and $\mathscr{L}^d$ described above is. The answer is in the following theorem.

Theorem 3 *Let $\mathscr{P}$ be a pseudo-Kleene algebra with a fix-point c. Then $\mathscr{P}$ is equal to the composition of $(c]$ and $[c)$ if and only if x, x' are comparable for each element $x \in P$.*

Proof Assume that x, x' are comparable for each $x \in P$. Since c is a fix-point, we have $c = c'$. Without loss of generality assume $x \leq x'$. Then

$$c = c \wedge c' \leq x \vee x' = x'$$

and hence also $x \leq c' = c$. Thus for each $x \in P$ we have either $x \in (c]$ or $x \in [c)$ proving that $\mathscr{P}$ is the composition of $(c]$ and $[c)$. Conversely, if $\mathscr{P}$ is the composition of $(c]$ and $[c)$ then every $x \in P$ belongs either to $(c]$ or $[c)$. Assume e.g. $x \in (c]$. Then $x \leq c$ thus $c = c' \leq x'$ whence $x \leq x'$. Thus x, x' are comparable for each $x \in P$. □

The following example shows that the condition of comparability x, x' does not follow from (1), (2) and (3).

Example 3 Consider the pseudo-Kleene algebra $\mathscr{P}$ depicted in Fig. 5.

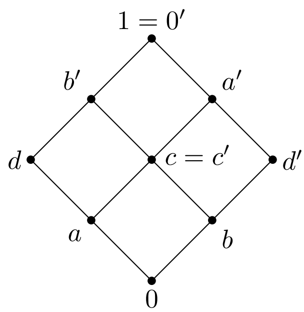

Fig. 5

Since it is a distributive lattice, it is in fact a Kleene algebra thus it satisfies (1), (2) and (3). Its fix-point is c. However d and d' are incomparable and one can see that $\mathscr{P}$ is not the composition od $(c]$ and $[c)$.

The previous theorem also yields that if $\mathscr{P}$ is composed of $\mathscr{L}$ and its dual then every congruence $\Theta \in \operatorname{Con}\mathscr{L}$ can be extended onto the whole $\mathscr{P}$ because Θ induces a congruence of $\mathscr{L}^d$, thus the relation which is union of Θ and the corresponding congruence on $\mathscr{L}^d$ is a congruence Ψ on the whole $\mathscr{P}$. Since the congruence classes of every $\Psi \in \operatorname{Con}\mathscr{P}$ are convex, also every congruence Ψ on $\mathscr{P}$ can be decomposed in a congruence $\Theta_1 = \Psi|_{\mathscr{L}}$ and $\Theta_2 = \Psi|_{\mathscr{L}^d}$. This immediately yields the following result.

Theorem 4 *Let $\mathscr{P}$ be a pseudo-Kleene algebra with a fix-point c such that x, x' are comparable for each $x \in P$. Then $\mathscr{P}$ is subdirectly irreducible if and only if its sublattice $(c]$ is subdirectly irreducible.*

This can be illustrated by the following example.

Example 4 Since the lattice N_5 is subdirectly irreducible, the pseudo-Kleene algebra from Example 2 (visualized in Fig. 3) is subdirectly irreducible. Since the lattice M_3 is subdirectly irreducible, the pseudo-Kleene algebra $\mathscr{P}$ depicted in Fig. 6 is subdirectly irreducible. In fact, M_3 is a simple lattice; thus also $\mathscr{P}$ is simple.

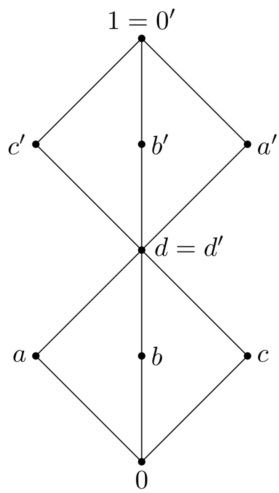

Fig. 6

Remark 1 Since the only subdirectly irreducible distributive lattice is the two-element one, the only subdirectly irreducible Kleene algebra with a fix-point is the three element chain $0 < c < 1$, where $0' = 1$, $1' = 0$, and $c = c'$. This is the result of [4].

An example of subdirectly irreducible pseudo-Kleene algebra which is not distributive and has no fix-point is visualized in Fig. 7. This is in fact a simple pseudo-Kleene algebra.

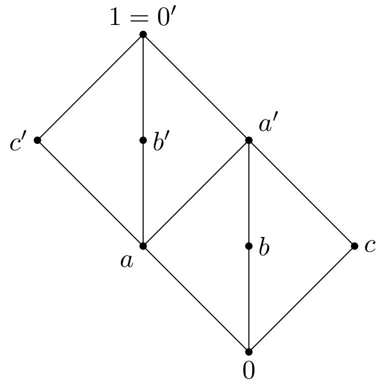

Fig. 7

Using this example, we can extend the construction producing pseudo-Kleene algebras without fix-points as the composition. Consider a lattice $\mathscr{L}$ with the top element 1 and let $[a, 1]$ be a filter of $\mathscr{L}$. Denote by x^* the elements of its dual $\mathscr{L}^d$. Then $[1^*, a^*]$ is an ideal of $\mathscr{L}^d$. If this filter $[a, 1]$ can be organized into a pseudo-Kleene algebra by a suitable antitone involution, then $\mathscr{L}$ and $\mathscr{L}^d$ can be glued together by identifying $[a, 1]$ with $[1^*, a^*]$, and the resulting lattice with an antitone involution is a pseudo-Kleene algebra. If a pseudo-Kleene algebra $\mathscr{P}$ is produced in this way, it is easy to show that $\mathscr{P}$ is subdirectly irreducible if and only if $\mathscr{L}$ is a subdirectly irreducible lattice. Hence, the pseudo-Kleene algebra visualized in Fig. 6 is subdirectly irreducible.

Acknowledgement This research is supported by the bilateral project I 1923--N25 of the Austrian Science Fund (FWF) and 15-34697L of the Czech Science Foundation (GAČR), and by the Palacký University project IGA PrF 2015010.

References

[1] Birkhoff, G.: Lattice Theory. Amer. Math. Soc. Colloqnium Publ. **25**, second edition, *American Mathematical Society*, New York, 1948.

[2] Cignoli, R.: *Injective de Morgan and Kleene algebras.* Proc. Amer. Math. Soc. **47** (1975), 269–278.

[3] Crvenkovič, S., Madarasz, R. Sz.: *On Kleene algebras.* Theoret. Comput. Sci. **108** (1993), 17–24.

[4] Kalman, J. A.: *Lattices with involution.* Trans. Amer. Math. Soc. **87** (1958), 485–491.