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Study of Stability in Nonlinear Neutral Differential Equations with Variable Delay Using Krasnoselskii–Burton's Fixed Point

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Abstract

In this paper, we use a modification of Krasnoselskii's fixed point theorem introduced by Burton (see [6] Theorem 3) to obtain stability results of the zero solution of the totally nonlinear neutral differential equation with variable delay

$$x'(t) = -a(t)h(x(t)) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))).$$

The stability of the zero solution of this eqution provided that $h(0) = Q(t,0) = G(t,0,0) = 0$. The Caratheodory condition is used for the functions Q and G.

Key words: Fixed point, stability, delay, stability, nonlinear neutral equation, large contraction mapping, integral equation.

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1 Introduction

More than 100 years ago, the world famous mathematician Lyapunov established the Lyapunov direct method to study stability problems. From then on, Lyapunov's direct method was widely used to study the stability of solutions of ordinary differential equations and functional differential equations, see e.g. [5]–[10], [12, 22], [26]–[29] and the references therein. But the expressions of Lyapunov functional are very complicated and hard to construct.

Recently, many authors have realized that the fixed points theory can be used to study the stability of the solution. Becker, Furumochi, Zhang and Burton considered the differential equation (see [1]–[4], [10, 11], [13]–[21] and [24]). The most striking object is that the fixed point method does not only solve the problem on stability but has a significant advantage over Liapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [7]). While it remains an art to construct a Liapunov's functional when it exists, a fixed point method, in one step, yields existence, uniqueness and stability. All we need, to use the fixed point method, is a complete metric space, a suitable fixed point theorem and an elementary integral methods to solve problems that have frustrated investigators for decades.

Recently, in [4], the Krasnoselskii–Burton's fixed point theorem was used to establish the stability and asymptotic stability of the zero solution for the first-order nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)h(x(t)) + c(t)x'(t-\tau(t)) + b(t)G(x(t), x(t-\tau(t))). \tag{1.1}$$

In [21], the authors used Krasnoselskii's fixed point theorem to establish the existence of periodic solutions for the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))). \tag{1.2}$$

Also, the authors used the contraction mapping principle to show the uniqueness of periodic solutions and stability of the zero solutions of (1.2).

This paper is mainly concerned with the stability and asymptotic stability of the zero solution of the nonlinear neutral differential equation with functional delay expressed as follows

$$\frac{d}{dt}x(t) = -a(t)h(x(t)) + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))), \tag{1.3}$$

with an assumed initial function

$$x(t) = \psi(t),\ t \in [m_0, 0],$$

where $\psi \in C([m_0, 0], \mathbb{R})$, $m_0 = \inf\{t - \tau(t) : t \geq 0\}$. Throughout this paper we assume that $a \in C(\mathbb{R}^+, \mathbb{R})$ with $a > 0$, $a \in L^1[0, \infty)$ is bounded, $\tau \in C(\mathbb{R}^+, \mathbb{R})$ and $h\colon \mathbb{R} \to \mathbb{R}$ is continuous, $Q\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory condition. Our purpose here is to use a modification of

Krasnoselskii's fixed point theorem due Burton (see [6], Theorem 3) to show the stability and asymptotic stability of the zero solution of equation (1.3). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [6], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.3) into an integral equation written as a sum of two mappings, one is a large contraction and the other is completely continuous. After that, we use a variant of Krasnoselskii's fixed point theorem, to show the stability and asymptotic stability of the zero solution.

Note that in our consideration the neutral term $\frac{d}{dt}Q(t, x(t-\tau(t)))$ of (1.3) produces nonlinearity in the derivative term $\frac{d}{dt}x(t-\tau(t))$. The neutral term $\frac{d}{dt}x(t-\tau(t))$ of (1.1) in [4] enters linearly. As a consequence, our analysis is different form that in [4].

The organization of this paper is as follows. In Section 2, we present the inversion of nonlinear neutral differential equation (1.3), some definitions and Krasnoselskii–Burton's fixed point theorem. For details on Krasnoselskii–Burton's theorem we refer the reader to [6]. In Sections 3, we present our main results on stability of the zero solutions of (1.3).

2 Preliminaries

We begin this section by the following Lemma.

Lemma 1. *x is a solution of (1.3) if and only if*

$$\begin{aligned} &x(t) \\ &= [\psi(0) - Q(0, \psi(-\tau(0)))]\, e^{-\int_0^t a(u)\,du} \\ &+ \int_0^t a(s) e^{-\int_s^t a(u)\,du} H(x(s))\,ds + Q(t, x(t-\tau(t))) \\ &+ \int_0^t e^{-\int_s^t a(u)\,du} \left[-a(s)Q(s, x(s-\tau(s))) + G(s, x(s), x(s-\tau(s)))\right] ds, \end{aligned} \tag{2.1}$$

where

$$H(x) = x - h(x). \tag{2.2}$$

Proof. Let x be a solution of (1.3). Rewrite the equation (1.3) as

$$\begin{aligned} &\frac{d}{dt}\left[x(t) - Q(t, x(t-\tau(t)))\right] + a(t)\left[x(t) - Q(t, x(t-\tau(t)))\right] \\ &= a(t)\left[x(t) - Q(t, x(t-\tau(t)))\right] - a(t)h(x(t)) \\ &+ G(t, x(t), x(t-\tau(t)) \\ &= a(t)\left[x(t) - h(x(t))\right] \\ &+ G(t, x(t), x(t-\tau(t))) - a(t)Q(t, x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $\exp\left(\int_0^t a(u)\,du\right)$ and then integrate from 0 to t, we obtain

$$\begin{aligned}
&\int_0^t \left[[x(t) - Q(t, x(t-\tau(t)))]\, e^{\int_0^s a(u)\,du}\right]' ds \\
&= \int_0^t a(s)\,[x(s) - h(x(s))]\, e^{\int_0^s a(u)\,du}\, ds \\
&+ \int_0^t [G(s, x(s), x(s-\tau(s))) - a(s)Q(s, x(s-\tau(s)))]\, e^{\int_0^s a(u)\,du}\, ds.
\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&[x(t) - Q(t, x(t-\tau(t)))]\, e^{\int_0^t a(u)\,du} - \psi(0) + Q(0, \psi(-\tau(0))) \\
&= \int_0^t a(s)\,[x(s) - h(x(s))]\, e^{\int_0^s a(u)\,du}\, ds \\
&+ \int_0^t [G(s, x(s), x(s-\tau(s))) - a(s)Q(s, x(s-\tau(s)))]\, e^{\int_0^s a(u)\,du}\, ds.
\end{aligned}$$

By dividing both sides of the above equation by $\exp\left(\int_0^t a(u)du\right)$ we obtain

$$\begin{aligned}
&x(t) - Q(t, x(t-\tau(t))) - [\psi(0) - Q(0, x(-\tau(0)))]\, e^{-\int_0^t a(u)\,du} \\
&= \int_0^t a(s)\,[x(s) - h(x(s))]\, e^{-\int_s^t a(u)\,du}\, ds \\
&+ \int_0^t [G(s, x(s), x(s-\tau(s))) - a(s)Q(s, x(s-\tau(s)))]\, e^{-\int_s^t a(u)\,du}\, ds. \qquad (2.3)
\end{aligned}$$

The converse implication is easily obtained and the proof is complete. □

Now, we give some definitions which will be used in this paper.

Definition 1. The map $f\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Carathéodory conditions with respect to $L^1[0,\infty)$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Lebesgue measurable.

(ii) For almost all $t \in [0,\infty)$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0,\infty), \mathbb{R}^+)$ such that for almost all $t \in [0,\infty)$ and for all z such that $|z| < r$, we have $|f(t,z)| \leq \alpha_r(t)$.

T. A. Burton studied the theorem of Krasnoselskii (see [7, 25]) and observed (see [5, 11]) that Krasnoselskii's result can be more interesting in applications with certain changes and formulated Theorem 1 below (see [5] for its proof).

Definition 2. Let $(\mathcal{M}, d)$ be a metric space and assume that $B\colon \mathcal{M} \to \mathcal{M}$. B is said to be a large contraction, if for $\varphi, \phi \in \mathcal{M}$, with $\varphi \neq \phi$, we have $d(B\varphi, B\phi) < d(\varphi, \phi)$, and if $\forall \epsilon > 0$, $\exists \delta < 1$ such that

$$[\varphi, \phi \in \mathcal{M},\ d(\varphi,\phi) \geq \epsilon] \Longrightarrow d(B\varphi, B\phi) < \delta d(\varphi, \phi).$$

It is proved in [6] that a large contraction defined on a bounded and complete metric space has a unique fixed point.

Theorem 1. *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{X}, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that*

(i) A is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$,

(ii) B is large contraction,

(iii) $x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$,

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

Here we manipulate function spaces defined on infinite t-intervals. So, for compactness we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([7], Theorem 1.2.2 p. 20) and is as follows.

Theorem 2. *Let $q\colon \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}^+$ with $|\varphi_n(t)| \le q(t)$ for $t \in \mathbb{R}^+$, then there is a subsequence that converges uniformly on $\mathbb{R}^+$ to a continuous function $\varphi(t)$ with $|\varphi(t)| \le q(t)$ for $t \in \mathbb{R}^+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

3 Main results

From the existence theory, which can be found in [7] or [23], we conclude that for each continuous initial function $\psi \in C([m_0, 0], \mathbb{R})$, there exists a continuous solution $x(t, 0, \psi)$ which satisfies (1.3) on an interval $[0, \sigma)$ for some $\sigma > 0$ and $x(t, 0, \psi) = \psi(t)$, $t \in [m_0, 0]$. We refer the reader to [7] for the stability definitions.

To apply Theorem 1, we need to define a Banach space $\mathcal{X}$, a closed bounded convex subset $\mathcal{M}$ of $\mathcal{X}$ and construct two mappings; one large contraction and the other is compact operator. So, let $w\colon [m_0, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $w(m_0) = 1$, $w(t) \to \infty$ as $t \to \infty$. Let $(\mathcal{S}, |\cdot|_w)$ be the Banach space of continuous $\varphi\colon [m_0, \infty) \to \mathbb{R}$ for which

$$|\varphi|_w = \sup_{t \in [m_0, \infty)} \left| \frac{\varphi(t)}{w(t)} \right| < \infty.$$

Let $R \in (0, 1]$ and define the set

$$\mathcal{M} := \{\varphi \in \mathcal{S} : \varphi \text{ is Lipschitzian}, \ |\varphi(t, 0, \psi)| \le R, \ t \in [m_0, \infty)\}. \qquad (3.1)$$

Clearly, if $\{\varphi_n\}$ is a sequence of l_1-Lipschitzian functions converging to some function φ, then

$$\begin{aligned} |\varphi(t) - \varphi(s)| &= |\varphi(t) - \varphi_n(t) + \varphi_n(t) - \varphi_n(s) + \varphi_n(s) - \varphi(s)| \\ &\le |\varphi(t) - \varphi_n(t)| + |\varphi_n(t) - \varphi_n(s)| + |\varphi_n(s) - \varphi(s)| \\ &\le l_1 |t - s|, \end{aligned}$$

as $n \to \infty$, which implies φ is l_1-Lipschitzian. It is clear that $\mathcal{M}$ is closed convex and bounded. For $\varphi \in \mathcal{M}$ and $t \geq 0$, we define by (2.1) the mapping $\mathcal{P}\colon \mathcal{M} \to \mathcal{S}$ as follows:

$$\begin{aligned}
&(\mathcal{P}\varphi)(t) \\
&= [\psi(0) - Q(0, \psi(-\tau(0)))] \, e^{-\int_0^t a(u)\,du} \\
&+ \int_0^t a(s) e^{-\int_s^t a(u)\,du} H(\varphi(s))\,ds + Q(t, \varphi(t-\tau(t))) \\
&+ \int_0^t e^{-\int_s^t a(u)\,du} \left[G(s, \varphi(s), \varphi(s-\tau(s))) - a(s) Q(s, \varphi(s-\tau(s))) \right] ds. \qquad (3.2)
\end{aligned}$$

Therefore, we express mapping (3.2) as

$$\mathcal{P}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}, \mathcal{B}\colon \mathcal{M} \to \mathcal{S}$ are given by

$$\begin{aligned}
(\mathcal{A}\varphi)(t) &= Q(t, \varphi(t-\tau(t))) \\
&\quad + \int_0^t e^{-\int_s^t a(u)\,du} \left[G(s, \varphi(s), \varphi(s-\tau(s))) - a(s) Q(s, \varphi(s-\tau(s))) \right] ds,
\end{aligned} \qquad (3.3)$$

and

$$\begin{aligned}
(\mathcal{B}\varphi)(t) &= [\psi(0) - Q(0, \psi(-\tau(0)))] \, e^{-\int_0^t a(u)\,du} \\
&\quad + \int_0^t a(s) e^{-\int_s^t a(u)\,du} H(\varphi(s))\,ds.
\end{aligned} \qquad (3.4)$$

By applying Theorem 1, we need to prove that $\mathcal{P}$ has a fixed point φ on the set $\mathcal{M}$, where $\varphi(t) = x(t, 0, \psi)$ for $t \geq 0$ and $x(t, 0, \psi) = \psi(t)$ on $[m_0, 0]$, $x(t, 0, \psi)$ satisfies (1.3) and $|\varphi(t, 0, \psi)| \leq R$ with $R \in (0, 1]$. For $t \geq 0$, we will assume that the following conditions hold:
The function Q is locally Lipschitz continuous, then for $t \geq 0$ and $x, y \in \mathcal{M}$ there exist a constants $E_Q > 0$, such that

$$|Q(t, x) - Q(t, y)| \leq E_Q \|x - y\|. \qquad (3.5)$$

The functions Q and G satisfy Carathéodory conditions with respect to $L^1[0, \infty)$, such that

$$|Q(t, \varphi(t-\tau(t)))| \leq q_R(t) \leq \frac{\alpha_1}{2} R, \qquad (3.6)$$

$$|G(t, \varphi(t), \varphi(t-\tau(t)))| \leq g_{\sqrt{2}R}(t) \leq \alpha_2 a(t) R, \qquad (3.7)$$

$$J(\alpha_1 + \alpha_2) \leq 1, \qquad (3.8)$$

where α_i, $i = 1, 2$ are positive constants and $J > 3$. Now, assume that there are constants $l_2, l_3 > 0$ such that for $0 \leq t_1 < t_2$

$$|\tau(t_2) - \tau(t_1)| \leq l_2 \left| t_2 - t_1 \right|, \qquad (3.9)$$

$$\left|\int_{t_1}^{t_2} a(u)\,du\right| \leq l_3 \left|t_2 - t_1\right|. \tag{3.10}$$

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) in Theorem 1.

Lemma 2. *For $\mathcal{A}$ defined in (3.3), suppose that (3.5)–(3.10) hold. Then, $\mathcal{A}\colon \mathcal{M} \to \mathcal{M}$ and $\mathcal{A}$ is continuous and $\mathcal{A}\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$.*

Proof. Let $\mathcal{A}$ be defined by (3.3). Then, for any $\varphi \in \mathcal{M}$, we have

$$\begin{aligned}
|\mathcal{A}\varphi(t)| &\leq |Q(t, \varphi(t-\tau(t)))| + \int_0^t e^{-\int_s^t a(u)\,du}\left[a(s)\left|Q(s, \varphi(s-\tau(s)))\right|\right.\\
&\quad \left.+ \left|G(s, \varphi(s), \varphi(s-\tau(s)))\right|\right] ds \\
&\leq q_R(t) + R\int_0^t e^{-\int_s^t a(u)\,du}\left(a(s)\frac{q_R(s)}{R} + \frac{g_{\sqrt{2}R}(s)}{R}\right) ds \\
&\leq \frac{\alpha_1}{2}R + \frac{\alpha_1}{2}R + \alpha_2 R \leq \frac{R}{J} < R.
\end{aligned}$$

That is $|\mathcal{A}\varphi(t)| < R$. Second we show that, for any $\varphi \in \mathcal{M}$ the function $\mathcal{A}\varphi$ is Lipschitzian. Let $\varphi \in \mathcal{M}$, and let $0 \leq t_1 < t_2$, then

$$\begin{aligned}
&|\mathcal{A}\varphi(t_2) - \mathcal{A}\varphi(t_1)| \\
&\leq |Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))| \\
&+ \left|\int_0^{t_2} e^{-\int_s^{t_2} a(u)\,du}\left[-a(s)Q(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right] ds\right. \\
&- \left.\int_0^{t_1} e^{-\int_s^{t_1} a(u)\,du}\left[-a(s)Q(s, \varphi(s-\tau(s))) + G(s, \varphi(s), \varphi(s-\tau(s)))\right] ds\right|.
\end{aligned} \tag{3.11}$$

By hypotheses (3.5), (3.9) and (3.10), we have

$$\begin{aligned}
&\left|Q\left(t_2, \varphi\left(t_2 - \tau\left(t_2\right)\right)\right) - Q\left(t_1, \varphi\left(t_1 - \tau\left(t_1\right)\right)\right)\right| \\
&\leq E_Q l_1 \left|\left(t_2 - t_1\right) - \left(\tau\left(t_2\right) - \tau\left(t_1\right)\right)\right| \\
&\leq \left(E_Q l_1 + E_Q l_1 l_2\right)\left|t_2 - t_1\right|,
\end{aligned} \tag{3.12}$$

where l_1 is the Lipschitz constant of φ. In the same way, by (3.6), (3.7) and

(3.10), we have

$$\begin{aligned}
&\left|\int_0^{t_2} e^{-\int_s^{t_2} a(u)\,du}\left[-a(s)Q(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]ds\right.\\
&\left.-\int_0^{t_1} e^{-\int_s^{t_1} a(u)\,du}\left[-a(s)Q(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]ds\right|\\
&\leq\left|\int_0^{t_1}\left[-a(s)Q(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]\right.\\
&\left.\times\, e^{-\int_s^{t_1} a(u)\,du}\left(e^{-\int_{t_1}^{t_2} a(u)\,du}-1\right)ds\right|\\
&+\left|\int_{t_1}^{t_2} e^{-\int_s^{t_2} a(u)\,du}\left[-a(s)Q(s,\varphi(s-\tau(s)))+G(s,\varphi(s),\varphi(s-\tau(s)))\right]ds\right|\\
&\leq\left(\frac{\alpha_1}{2}+\alpha_2\right)R\left|e^{-\int_{t_1}^{t_2} a(u)\,du}-1\right|\int_0^{t_1} a(s)e^{-\int_s^{t_1} a(u)\,du}\,ds\\
&+\int_{t_1}^{t_2} e^{-\int_s^{t_2} a(u)\,du}\left(g_{\sqrt{2}R}(s)+a(s)q_R(s)\right)ds\\
&\leq\left(\frac{\alpha_1}{2}+\alpha_2\right)R\int_{t_1}^{t_2} a(s)\,ds\\
&+\int_{t_1}^{t_2} a(s)e^{-\int_s^{t_2} a(u)\,du}d\left(\int_{t_1}^{s}\left(g_{\sqrt{2}R}(r)+a(r)q_R(r)\right)dr\right)ds\\
&\leq\left(\frac{\alpha_1}{2}+\alpha_2\right)R\int_{t_1}^{t_2} a(s)\,ds+\left[e^{-\int_s^{t_2} a(u)\,du}\int_{t_1}^{s}\left(g_{\sqrt{2}R}(r)+a(r)q_R(r)\right)dr\right]_{t_1}^{t_2}\\
&+\int_{t_1}^{t_2} a(s)e^{-\int_s^{t_2} a(u)\,du}\int_{t_1}^{s}\left(g_{\sqrt{2}R}(r)+a(r)q_R(r)\right)drds\\
&\leq\left(\frac{\alpha_1}{2}+\alpha_2\right)R\int_{t_1}^{t_2} a(s)\,ds\\
&+\int_{t_1}^{t_2}\left(g_{\sqrt{2}R}(s)+a(s)q_R(s)\right)ds\left(1+\int_{t_1}^{t_2} a(s)e^{-\int_s^{t_2} a(u)\,du}\,ds\right)\\
&\leq\left(\frac{\alpha_1}{2}+\alpha_2\right)R\int_{t_1}^{t_2} a(s)\,ds+2\int_{t_1}^{t_2}\left(g_{\sqrt{2}R}(s)+a(s)q_R(s)\right)ds\\
&\leq\left(\frac{\alpha_1}{2}+\alpha_2\right)R\int_{t_1}^{t_2} a(s)\,ds+2\left(\frac{\alpha_1}{2}+\alpha_2\right)R\int_{t_1}^{t_2} a(s)\,ds\\
&\leq 3R\left(\frac{\alpha_1}{2}+\alpha_2\right)l_3\left|t_2-t_1\right|.
\end{aligned}\tag{3.13}$$

Thus, by substituting (3.12) and (3.13) in (3.11), we obtain

$$\begin{aligned}
\left|\mathcal{A}\varphi(t_2)-\mathcal{A}\varphi(t_1)\right| &\leq\left(E_Q l_1+E_Q l_1 l_2\right)\left|t_2-t_1\right|+3R\left(\frac{\alpha_1}{2}+\alpha_2\right)l_3\left|t_2-t_1\right|\\
&=K\left|t_2-t_1\right|,
\end{aligned}$$

for some constant $K > 0$. This shows that $\mathcal{A}\varphi$ is Lipschitzian if φ is. This completes the proof of $\mathcal{A} : \mathcal{M} \to \mathcal{M}$.

Since $\mathcal{A}\varphi$ is Lipschitzian, then $\mathcal{A}\mathcal{M}$ is equicontinuous, which implies that the set $\mathcal{A}\mathcal{M}$ resides in a compact set in the space $(\mathcal{S}, |\cdot|_w)$.

Now, we show that $\mathcal{A}$ is continuous in the weighted norm, let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&\left|\frac{\mathcal{A}\varphi_n(t) - \mathcal{A}\varphi(t)}{w(t)}\right| \\
&\le |Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))|_w \\
&+ \int_0^t a(s) e^{-\int_s^t a(u)\,du} |Q(s, \varphi_n(s-\tau(s))) - Q(s, \varphi(s-\tau(s)))|_w \, ds \\
&+ \int_0^t e^{-\int_s^t a(u)\,du} |G(s, \varphi_n(s), \varphi_n(s-\tau(s))) - G(s, \varphi(s), \varphi(s-\tau(s)))|_w \, ds.
\end{aligned}$$

By the dominated convergence theorem, $\lim_{n\to\infty} |(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)|_w = 0$. Then $\mathcal{A}$ is continuous. This completes the proof of $\mathcal{A} \colon \mathcal{M} \to \mathcal{M}$ is continuous and $\mathcal{A}\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$. □

Now, we state an important result implying that the mapping H given by (2.2) is a large contraction on the set $\mathcal{M}$. This result was already obtained in [1, Theorem 3.4] and for convenience we present below its proof. We shall assume that

(H1) $h \colon \mathbb{R} \to \mathbb{R}$ is continuous on $[-R, R]$ and differentiable on $(-R, R)$,

(H2) The function h is strictly increasing on $[-R, R]$,

(H3) $\sup_{t\in(-R,R)} h'(t) \le 1$.

Theorem 3. *Let $h \colon \mathbb{R} \to \mathbb{R}$ be a function satisfying (H1)–(H3). Then the mapping H in (2.2) is a large contraction on the set $\mathcal{M}$.*

Proof. Let $\varphi, \phi \in \mathcal{M}$ with $\varphi \ne \phi$. Then $\varphi(t) \ne \phi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\varphi, \phi)$, i.e.,

$$D(\varphi, \phi) = \{t \in \mathbb{R} : \varphi(t) \ne \phi(t)\}.$$

For all $t \in D(\varphi, \phi)$, we have

$$\begin{aligned}
|(H\varphi)(t) - (H\phi)(t)| &\le |\varphi(t) - \phi(t) - h(\varphi(t)) + h(\phi(t))| \\
&\le |\varphi(t) - \phi(t)| \left|1 - \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)}\right|.
\end{aligned} \tag{3.14}$$

Since h is a strictly increasing function we have

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} > 0 \text{ for all } t \in D(\varphi, \phi). \tag{3.15}$$

For each fixed $t \in D(\varphi, \phi)$ define the interval $I_t \subset [-R, R]$ by

$$I_t = \begin{cases} (\varphi(t), \phi(t)) \text{ if } \varphi(t) < \phi(t), \\ (\phi(t), \varphi(t)) \text{ if } \phi(t) < \varphi(t). \end{cases}$$

The mean value theorem implies that for each fixed $t \in D(\varphi, \phi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = h'(c_t).$$

By (H2), (H3) we have

$$0 \leq \inf_{s \in (-R,R)} h'(s) \leq \inf_{s \in I_t} h'(s) \leq h'(c_t) \leq \sup_{s \in I_t} h'(s) \leq \sup_{s \in (-R,R)} h'(s) \leq 1. \tag{3.16}$$

Hence, by (3.14)–(3.16) we obtain

$$|(H\varphi)(t) - (H\phi)(t)| \leq |\varphi(t) - \phi(t)| \left| 1 - \inf_{s \in (-R,R)} h'(s) \right|, \tag{3.17}$$

for all $t \in D(\varphi, \phi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\epsilon \in (0, 1)$ and assume that φ and ϕ are two functions in $\mathcal{M}$ satisfying

$$\epsilon \leq \sup_{t \in (-R,R)} |\varphi(t) - \phi(t)| = \|\varphi - \phi\|.$$

If $|\varphi(t) - \phi(t)| \leq \frac{\epsilon}{2}$ for some $t \in D(\varphi, \phi)$, then we get by (3.16) and (3.17) that

$$|(H\varphi)(t) - (H\phi)(t)| \leq \frac{1}{2} |\varphi(t) - \phi(t)| \leq \frac{1}{2} \|\varphi - \phi\|. \tag{3.18}$$

Since h is continuous and strictly increasing, the function $h\left(s + \frac{\epsilon}{2}\right) - h(s)$ attains its minimum on the closed and bounded interval $[-R, R]$. Thus, if $\frac{\epsilon}{2} \leq |\varphi(t) - \phi(t)|$ for some $t \in D(\varphi, \phi)$, then by (H2) and (H3) we conclude that

$$1 \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2R} \min\left\{ h\left(s + \frac{\epsilon}{2}\right) - h(s) : s \in [-R, R] \right\} > 0.$$

Hence, (3.14) implies

$$|(H\varphi)(t) - (H\phi)(t)| \leq (1 - \lambda) \|\varphi - \phi\|. \tag{3.19}$$

Consequently, combining (3.18) and (3.19) we obtain

$$|(H\varphi)(t) - (H\phi)(t)| \leq \delta \|\varphi - \phi\|, \tag{3.20}$$

where

$$\delta = \max\left\{ \frac{1}{2}, 1 - \lambda \right\}.$$

The proof is complete. □

The next result shows the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions, for this assume that

$$\max\{|H(-R)|, |H(R)|\} \leq \frac{2R}{J}. \tag{3.21}$$

Choose $\gamma > 0$ small enough such that

$$[1 + E_Q]\gamma e^{-\int_0^t a(u)\,du} + \frac{R}{J} + \frac{2R}{J} \leq R. \tag{3.22}$$

The chosen in the relation (3.22) will be used below in Lemma 3 and Theorem 4 to show that if $\epsilon = R$ and if $\|\psi\| < \gamma$, then the solutions satisfies $|x(t, 0, \psi)| < \epsilon$.

Lemma 3. *Let $\mathcal{B}$ be defined by (3.4), suppose (3.9), (3.10), (H1)–(H3), (3.21) and (3.22) hold. Then $\mathcal{B}\colon \mathcal{M} \to \mathcal{M}$ and $\mathcal{B}$ is a large contraction.*

Proof. Let $\mathcal{B}$ be defined by (3.4). Obviously, $\mathcal{B}$ is continuous with the weighted norm. Let $\varphi \in \mathcal{M}$

$$\begin{aligned}
|\mathcal{B}\varphi(t)| &\leq |\psi(0) - Q(0, \psi(-\tau(0)))| e^{-\int_0^t a(u)\,du} \\
&\quad + \int_0^t a(s) e^{-\int_s^t a(u)\,du} |H(\varphi(s))|\,ds \\
&\leq [1 + E_Q]\gamma e^{-\int_0^t a(u)\,du} \\
&\quad + \int_0^t a(s) e^{-\int_s^t a(u)\,du} \max\{|H(-R)|, |H(R)|\}\,ds < R,
\end{aligned}$$

and we use a method like in Lemma 2, we deduce that, for any $\varphi \in \mathcal{M}$ the function $\mathcal{B}\varphi$ is Lipschitzian, which implies $\mathcal{B}\colon \mathcal{M} \to \mathcal{M}$.

By Theorem 3, H is large contraction on $\mathcal{M}$, then for any $\varphi, \phi \in \mathcal{M}$, with $\varphi \neq \phi$ and for any $\epsilon > 0$, from the proof of that Theorem, we have found a $\delta < 1$, such that

$$\left|\frac{\mathcal{B}\varphi(t) - \mathcal{B}\phi(t)}{w(t)}\right| \leq \int_0^t a(s) e^{-\int_s^t a(u)\,du} \left|H(\varphi(u)) - H(\phi(u))\right|_w du \leq \delta|\varphi - \phi|_w.$$

The proof is complete. □

Theorem 4. *Assume the hypothesis of Lemmas 2 and 3. Let $\mathcal{M}$ defined by (3.1). Then the equation (1.3) has a solution in $\mathcal{M}$.*

Proof. By Lemmas 2, 4, $\mathcal{A} : \mathcal{M} \to \mathcal{M}$ is continuous and $\mathcal{A}(\mathcal{M})$ is contained in a compact set. Also, from Lemma 3, the mapping $\mathcal{B} : \mathcal{M} \to \mathcal{M}$ is a large contraction. Next, we show that if $\varphi, \phi \in \mathcal{M}$, we have $\|\mathcal{A}\varphi + \mathcal{B}\phi\| \leq R$. Let $\varphi, \phi \in \mathcal{M}$ with $\|\varphi\|, \|\phi\| \leq R$. By (3.6)–(3.8)

$$\begin{aligned}
\|\mathcal{A}\varphi + \mathcal{B}\varphi\| &\leq (1 + E_Q)\gamma e^{-\int_0^t a(u)\,du} + (\alpha_1 + \alpha_2)R + \frac{2R}{J} \\
&\leq (1 + E_Q)\,\gamma e^{-\int_0^t a(u)\,du} + \frac{R}{J} + \frac{2R}{J} \leq R.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii–Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 1 this fixed point is a solution of (1.3). Hence (1.3) is stable. □

Now, for the asymptotic stability, define $\mathcal{M}_0$ by

$$\begin{aligned}\mathcal{M}_0 \, : = \{\varphi \in \mathcal{S} : \varphi \text{ is Lipschitzian, } |\varphi(t,0,\psi)| \leq R, \ t \in [m_0,\infty), \\ \varphi(t) = \psi(t) \text{ if } t \in [m_0,0] \text{ and } |\varphi(t)| \to 0 \text{ as } t \to \infty\}\,. \end{aligned} \tag{3.23}$$

All of the calculations in the proof of Theorem 4 hold with $w(t) = 1$ when $|\cdot|_w$ is replaced by the supremum norm $\|\cdot\|$. Now, assume that

$$t - \tau(t) \to \infty \text{ as } t \to \infty \text{ and } \int_0^t a(s)\,ds \to \infty \text{ as } t \to \infty, \tag{3.24}$$

$$q_R(t) \to 0 \text{ as } t \to \infty, \tag{3.25}$$

$$\frac{g_{\sqrt{2}R}(t)}{a(t)} \to 0 \text{ as } t \to \infty. \tag{3.26}$$

Lemma 4. *Let (3.5)–(3.10) and (3.24)–(3.26) hold. Then, the operator $\mathcal{A}$ maps $\mathcal{M}$ into a compact subset of $\mathcal{M}$.*

Proof. First, we deduce by the Lemma 2 that $\mathcal{A}(\mathcal{M})$ is equicontinuous. Next, we notice that for arbitrary $\varphi \in \mathcal{M}$ we have

$$|\mathcal{A}\varphi(t)| \leq q_R(t) + \int_0^t e^{-\int_s^t a(u)\,du} a(s) \left(q_R(s) + \frac{g_{\sqrt{2}R}(s)}{a(s)} \right) ds := q(t).$$

We see that $q(t) \to 0$ as $t \to \infty$ which implies that the set $\mathcal{A}\mathcal{M}$ resides in a compact set in the space $(\mathcal{S}, \|\cdot\|)$ by Theorem 2. □

Theorem 5. *Assume the hypothesis of Lemmas 2, 4 and 3 hold. Let $\mathcal{M}_0$ defined by (3.23). Then the equation (1.3) has a solution in $\mathcal{M}_0$.*

Proof. Note that, all of the steps in the proof of Theorem 4 hold with $w(t) = 1$ when $|\cdot|_w$ is replaced by the supremum norm $\|\cdot\|$. It is sufficient to show, for $\varphi \in \mathcal{M}_0$ then $\mathcal{A}\varphi \to 0$ and $\mathcal{B}\varphi \to 0$. Let $\varphi \in \mathcal{M}$ be fixed, we will prove that $|\mathcal{A}\varphi(t)| \to 0$ as $t \to \infty$, as above we have

$$\begin{aligned}|\mathcal{A}\varphi(t)| &\leq |Q(t,\varphi(t-\tau(t)))| \\ &+ \int_0^t e^{-\int_s^t a(t)\,du} \left[a(s)\,|Q(s,\varphi(s-\tau(s)))| + |G(s,\varphi(s),\varphi(s-\tau(s)))|\right] ds.\end{aligned}$$

First, we have

$$|Q(t,\varphi(t-\tau(t)))| \leq q_R(t) \to 0 \text{ as } t \to \infty,$$

Second, let $\epsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))|, |\varphi(t)| < \epsilon$, for $t \geq T$. Then we have

$$
\begin{aligned}
&\int_0^t e^{-\int_s^t a(u)\,du}\left[a(s)\left|Q(s,\varphi(s-\tau(s)))\right| + \left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right] ds \\
&= e^{-\int_T^t a(u)\,du} \\
&\times \int_0^T e^{-\int_s^T a(u)\,du}\left[a(s)\left|Q(s,\varphi(s-\tau(s)))\right| + \left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right] ds \\
&+ \int_T^t e^{-\int_s^t a(u)\,du}\left[a(s)\left|Q(s,\varphi(s-\tau(s)))\right| + \left|G(s,\varphi(s),\varphi(s-\tau(s)))\right|\right] ds \\
&\leq e^{-\int_T^t a(u)\,du}\left(\frac{\alpha_1}{2}+\alpha_2\right) R + \left(\frac{\alpha_1}{2}+\alpha_2\right)\epsilon.
\end{aligned}
$$

By (3.24) the term $e^{-\int_T^t a(u)du}\left(\frac{\alpha_1}{2}+\alpha_2\right) R$ is, as $t \to \infty$, arbitrarily small. In the same way, we obtain $\mathcal{B}\varphi \to 0$. This completes the proof. □

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