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Estimation in Connecting Measurements *

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Abstract

The aim of the paper is to show some possible statistical solutions of the connecting measurements. The algorithms were published in [1], [2] and [3]. The paper concentrates on numerical studies of these algorithms, finding estimators of parameters and comparing their covariance matrices.

Key words: Two stage regression models, uncertainty of the type A and B, BLUE.

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1 Introduction

We study two stage linear models, where we must respect uncertainty in connecting measurements and estimations of parameters for connecting measurements. We have got estimator $\widehat{\Theta}$ of parameter Θ in the first stage before measurements (we measure by an instrument with known parameters). In connection with uncertainty of estimation of parameters Θ for connected measurements we define "uncertainty of type B" in comparison with "uncertainty of type A", connected with accuracy of connecting measurements.

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We study the model where in the second stage (connecting measurements) occurs the constraints on parameters of the first and the second stage (type I).

We need to considered these constraints during finding estimators of parameters from the second stage.

We define $\mathcal{U}_\beta$ of unbiased estimators $\widetilde{\beta}$ of the parameters β in the regular model, where we respect errors in connecting points; and class $\widetilde{\mathcal{U}}_\beta$ of unbiased estimators $\widetilde{\widetilde{\beta}}$ of parameter β satisfying the constraints between parameters of the first and the second stage.

The estimators from the class $\mathcal{U}_\beta$ need not fulfil the constraints between parameters of the first and the second stages. There does not exist any jointly efficient estimator in the class $\widetilde{\mathcal{U}}_\beta$. Therefore we study estimators from the class $\widetilde{\mathcal{U}}_\beta$ which minimize a linear functional of the covariance matrix of the estimator $\widetilde{\widetilde{\beta}}$.

2 Estimation in model of connecting measurements with constraints of type I

Definition 2.1 The model of connecting measurement will be called random vector $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')$, with the mean values and the covariance matrix:

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \Theta \\ \beta \end{pmatrix}, \begin{pmatrix} \mathbf{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma}_{2,2} \end{pmatrix} \right],$$

where $\mathbf{X}_1, \mathbf{D}, \mathbf{X}_2$ are known $n_1 \times k_1, n_2 \times k_1, n_2 \times k_2$ matrices, with the condition $\mathcal{M}(\mathbf{D}') \subset \mathcal{M}(\mathbf{X}_1')$; Θ, β are unknown k_1 and k_2-dimensional vectors; $\mathbf{\Sigma}_{1,1}$ and $\mathbf{\Sigma}_{2,2}$ are known covariance matrices of vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

In this model the parameter Θ is estimated on the basis of the vector $\mathbf{Y}_1$ of the first stage and parameter β on the basis of vectors $\mathbf{Y}_2 - \mathbf{D}\hat{\Theta}$ and $\hat{\Theta}$. The results of measurements from the second stage (this means $\mathbf{Y}_2$) we cannot use for the change of the estimator $\hat{\Theta}$.

The parametric space of this model of connecting measurements $\mathbf{Y}$ according Definition 2.1 is

$$\underline{\Theta} = \{(\Theta', \beta') : \mathbf{B}\beta + \mathbf{C}\Theta + \mathbf{a} = \mathbf{0}\}$$

where $\mathbf{B}, \mathbf{C}$ are $q \times k_2, q \times k_1$ matrices and where $\mathbf{a}$ is q-dimensional vector, where $r(\mathbf{B}) = q < k_2$.

The vector Θ is the parameter of the first stage (connecting), the vector β is the parameter of the second stage (connected). In the second stage we have the unbiased estimator $\widehat{\Theta} = (\mathbf{X}_1 \mathbf{\Sigma}_{1,1}^{-1} \mathbf{X}_1)^{-1} \mathbf{X}_1' \mathbf{\Sigma}_{1,1}^{-1} \mathbf{Y}_1$ from the first stage and its covariance matrix $\mathrm{Var}(\widehat{\Theta}) = (\mathbf{X}_1 \mathbf{\Sigma}_{1,1}^{-1} \mathbf{X}_1)^{-1}$.

Definition 2.2 The model in Definition 2.1 in this parametric space $\underline{\Theta}$ is regular provided $r(\mathbf{X}_1) = k_1$, $r(\mathbf{X}_2) = k_2$, $\mathbf{\Sigma}_{1,1}, \mathbf{\Sigma}_{2,2}$ are positively definite matrices and $r(\mathbf{B}) = q$.

Definition 2.3 We will consider the model of connecting measurements according to Definition 2.1. Estimator $\mathbf{L}'\mathbf{Y}+\mathbf{d}$ of the function $f(\beta)=f'\beta$, where exists Θ where $\binom{\Theta}{\beta}\in\underline{\Theta}$, where f is given vector from $\mathcal{R}^k$ we call the best linear unbiased estimator (i.e. the best in the sense of variance) if it is

(i) unbiased: for all $(\Theta',\beta')\in\underline{\Theta}$ is $E(\mathbf{L}'Y+d)=f'\beta$,
(ii) efficient: $\mathrm{Var}(\mathbf{L}'\mathbf{Y}+d)\leq\mathrm{Var}(\widetilde{\mathbf{L}}'\mathbf{Y}+\widetilde{d})$, where $\widetilde{\mathbf{L}}'\mathbf{Y}+\widetilde{d}$ is arbitrary other unbiased estimator of function $f(\beta)$.

Lemma 2.1 *The class $\mathcal{U}_\beta$ of all linear unbiased estimators $\widetilde{\beta}$ of the parameter β based on the vectors $\mathbf{Y}_2-\mathbf{D}\hat{\Theta}$ and $\hat{\Theta}$ is*

$$\begin{aligned}\mathcal{U}_\beta=\big\{&[\mathbf{X}_2^-+\mathbf{Z}(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)+\mathbf{EBX}_2^-](\mathbf{Y}_2-\mathbf{D}\hat{\Theta})+\mathbf{EC}\hat{\Theta}+\mathbf{Ea}:\\ &\mathbf{Z}\text{ an arbitrary }k_2\times n_2\text{ matrix, }\mathbf{E}\text{ an arbitrary }k_2\times q\text{ matrix}\\ &\mathbf{X}_2^-\text{ an arbitrary but fixed }\mathbf{X}_2^-\in\mathcal{X}^-,\ ({}^-\text{means g-inverse})\big\}.\end{aligned}$$

Proof [1], p. 646.

Lemma 2.2 *The class $\widetilde{\mathcal{U}}_\beta$ of all linear unbiased estimators $\widetilde{\widetilde{\beta}}$ of the parameter β in the model from Definition 2.1 based on vectors $\mathbf{Y}_2-\mathbf{D}\hat{\Theta}$ and $\hat{\Theta}$, and satisfaying the (random) condition $\mathbf{B}\widetilde{\widetilde{\beta}}+\mathbf{C}\widehat{\Theta}+\mathbf{a}=\mathbf{0}$ is*

$$\begin{aligned}\widetilde{\mathcal{U}}_\beta=\big\{&[\mathbf{I}-\mathbf{B}^-\mathbf{B}]\big[\mathbf{X}_2^-+\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)+\mathbf{W}_2\mathbf{BX}_2^-\big](\mathbf{Y}_2-\mathbf{D}\widehat{\Theta})\\ &+\big[-\mathbf{B}^-+(\mathbf{I}-\mathbf{B}^-\mathbf{B})\mathbf{W}_2\big]\mathbf{C}\widehat{\Theta}+(\mathbf{I}-\mathbf{B}^-\mathbf{B})\mathbf{W}_2\mathbf{a}-\mathbf{B}^-\mathbf{a},\\ &\mathbf{W}_1\text{ an arbitrary }k_2\times n_2\text{ matrix,}\mathbf{W}_2\text{ an arbitrary }k_2\times q\text{ matrix}\\ &\mathbf{X}_2^-\text{ and }\mathbf{B}^-\text{ are arbitrary but fixed }\mathbf{X}_2^-\in\mathcal{X}^-,\mathbf{B}^-\in\mathcal{B}^-\text{ matrices}\big\}.\end{aligned}$$

Proof [1], p. 647.

Corollary 2.1 *Covariance matrix of the estimator $\widetilde{\widetilde{\beta}}$ is*

$$\begin{aligned}\mathrm{Var}(\widetilde{\widetilde{\beta}})=\ &(\mathbf{I}-\mathbf{B}^-\mathbf{B})\big[\mathbf{X}_2^-+\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)+\mathbf{W}_2\mathbf{BX}_2^-\big]\mathbf{\Sigma}_{2,2}\\ &\times\big[\mathbf{X}_2^-+\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)+\mathbf{W}_2\mathbf{BX}_2^-\big]'(\mathbf{I}-\mathbf{B}^-\mathbf{B})'\\ &+\{(\mathbf{I}-\mathbf{B}^-\mathbf{B})[-\mathbf{X}_2^-\mathbf{D}-\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)\mathbf{D}-\mathbf{W}_2\mathbf{BX}_2^-\mathbf{D}\\ &+\mathbf{W}_2\mathbf{C}]-\mathbf{B}^-\mathbf{C}\}]\mathbf{\Sigma}_{1,1}(\mathbf{I}-\mathbf{B}^-\mathbf{B})\\ &\times\{[-\mathbf{X}_2^-\mathbf{D}-\mathbf{W}_1(\mathbf{I}-\mathbf{X}_2\mathbf{X}_2^-)\mathbf{D}-\mathbf{W}_2\mathbf{BX}_2^-\mathbf{D}+\mathbf{W}_2\mathbf{C}]-\mathbf{B}^-\mathbf{C}\}'.\end{aligned}$$

Corollary 2.2 *Covariance matrix of the estimator $\widetilde{\widetilde{\beta}}$, for case of the model, where $\mathbf{X}_2=\mathbf{I}$, is*

$$\begin{aligned}\mathrm{Var}(\widetilde{\widetilde{\beta}})=\ &(\mathbf{I}-\mathbf{B}^-\mathbf{B})[\mathbf{I}+\mathbf{W}_2\mathbf{B}]\mathbf{\Sigma}_{2,2}\times[\mathbf{I}+\mathbf{W}_2\mathbf{B}]'(\mathbf{I}-\mathbf{B}^-\mathbf{B})'\\ &+\{(\mathbf{I}-\mathbf{B}^-\mathbf{B})[-\mathbf{D}-\mathbf{W}_2\mathbf{BD}+\mathbf{W}_2\mathbf{C}]-\mathbf{B}^-\mathbf{C}\}\\ &\times\mathbf{\Sigma}_{1,1}\{(\mathbf{I}-\mathbf{B}^-\mathbf{B})[-\mathbf{D}-\mathbf{W}_2\mathbf{BD}+\mathbf{W}_2\mathbf{C}]-\mathbf{B}^-\mathbf{C}\}'.\end{aligned}$$

Theorem 2.1 *In the class $\mathcal{U}_\beta$ in Lemma 2.1 (estimators $\widetilde{\beta}$ from $\mathcal{U}_\beta$ need not to satisfay condition $\mathbf{B}\widetilde{\beta}+\mathbf{C}\widehat{\Theta}+\mathbf{a}=\mathbf{0}$), there exists the jointly efficient estimator $\hat{\beta}^*$ of the vector β*

$$\hat{\beta}^* = \Big((\,\mathbf{X}_2',\ \mathbf{B}'\,)^-_{m(\mathbf{S})} \Big)' \begin{pmatrix} \mathbf{Y}_2 - \mathbf{D}\widehat{\Theta} \\ -\mathbf{C}\widehat{\Theta} - \mathbf{a} \end{pmatrix},$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11}, & \mathbf{S}_{12} \\ \mathbf{S}_{21}, & \mathbf{S}_{22} \end{pmatrix}$$

$$\begin{aligned} \mathbf{S}_{11} &= \mathbf{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}', & \mathbf{S}_{12} &= \mathbf{D}(\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}', \\ \mathbf{S}_{21} &= \mathbf{C}(\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}', & \mathbf{S}_{22} &= \mathbf{C}(\mathbf{X}_1'\mathbf{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'. \end{aligned}$$

Proof [1], p. 649.

Definition 2.4 The least squares estimator of the parameter β obtained under the condition $\mathbf{\Sigma}_{1,1} = \mathbf{0}$ ($\Rightarrow$ Var$(\widehat{\Theta}) = \mathbf{0}$) is called the standard estimator if in this estimator the vector Θ is substituted by $\widehat{\Theta}$.

Theorem 2.2 *The standard estimator $\hat{\beta}$ of the parameter β in the model according Definition 2.1 is given as*

$$\begin{aligned} \hat{\beta} = {} & (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta}) \\ & - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \\ & \times \{\mathbf{a} + \mathbf{C}\widehat{\Theta} + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta})\}, \end{aligned}$$

whereas this estimator is unbiased, it means $\mathbf{E}(\hat{\beta}) = \beta$.

Proof The best linear estimator $\hat{\beta}$ determined by the least squares method in the model $\mathbf{Y} \sim_n (\mathbf{D}\Theta + \mathbf{X}_2\beta, \mathbf{\Sigma}_{2,2})$ satisfying condition $\mathbf{B}\beta + \mathbf{C}\Theta + \mathbf{a} = \mathbf{0}$, where the parameter Θ is known, we get by minimizing the function

$$\begin{aligned} \phi(\beta) &= (\mathbf{Y}_2 - \mathbf{D}\Theta - \mathbf{X}_2\beta)'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta - \mathbf{X}_2\beta) - 2\lambda'[(\mathbf{a} + \mathbf{C}\Theta) + \mathbf{B}\beta] \\ &= (\mathbf{Y}_2 - \mathbf{D}\Theta)'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) - 2\beta'\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) + \beta'\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2\beta \\ &\quad - 2\lambda'[\mathbf{a} + \mathbf{C}\Theta + \mathbf{B}\beta]. \end{aligned}$$

We determine the derivative of the function $\phi(\beta)$

$$\frac{\partial\phi(\beta)}{\partial\beta} = -2\mathbf{X}_2\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) + 2\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2\hat{\beta} - 2\mathbf{B}'\lambda$$

and solve the system of equations

$$\begin{aligned} \frac{\partial\phi(\beta)}{\partial\beta} &= -2\mathbf{X}_2\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) + 2\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2\hat{\beta} - 2\mathbf{B}'\lambda = 0 \\ & B\beta + \mathbf{C}\Theta + \mathbf{a} = 0. \end{aligned}$$

From the first equation we get $\hat{\beta}$

$$\hat{\beta} = (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) + (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\lambda$$

and after substitution into the second equation

$$\mathbf{a} + \mathbf{C}\Theta + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'\lambda = \mathbf{0}$$

we determine

$$\lambda = -[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\{\mathbf{a} + \mathbf{C}\Theta + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta)\}.$$

After substitution λ into the first equation we get

$$\begin{aligned}
\hat{\beta} =\ & (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta) \\
& - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \\
& \times \{\mathbf{a} + \mathbf{C}\Theta + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta)\}, \\
\hat{\beta} =\ & -(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}(\mathbf{a} + \mathbf{C}\Theta) \\
& + \{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\
& \times (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{Y}_2 - \mathbf{D}\Theta), \\
\hat{\beta} =\ & \{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1} \\
& \times (\mathbf{Y}_2 - \mathbf{D}\Theta) - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\}(\mathbf{a} + \mathbf{C}\Theta).
\end{aligned}$$

By choosing $\widehat{\Theta}$ for Θ we get the standard estimator.

The assertion $\mathbf{E}(\hat{\beta}) = \beta$ is the result from our premise $\mathbf{E}(\widehat{\Theta}) = \Theta$ and the fact that $\mathbf{E}(Y_2) = \mathbf{D}\Theta + \mathbf{X}_2\beta$. Thus

$$\begin{aligned}
\mathbf{E}(\hat{\beta}) =\ & (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2\beta - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1} \\
& \{\mathbf{a} + \mathbf{C}\Theta + \mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2\beta\} = \beta,
\end{aligned}$$

because of $\mathbf{a} + \mathbf{C}\Theta + \mathbf{B}\beta = \mathbf{0}$. □

Theorem 2.3 *If* $\mathrm{Var}(\widehat{\Theta}) \neq \mathbf{0}$ *then the covariance matrix of the standard estimator* $\hat{\beta}$ *is formed by "uncertainty A" and "uncertainty B":*

$$\mathrm{Var}(\hat{\beta}) = \underbrace{\mathrm{Var}_0(\hat{\beta})}_{\substack{\textit{uncertainty}\\ \textit{type A}}} \underbrace{\begin{aligned}
& + \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\
& \times (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1} \\
& \times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle \\
& \times \mathrm{Var}(\widehat{\Theta}) \\
& \times \langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}\} \\
& \times (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1} \\
& \times \mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle'
\end{aligned}}_{\substack{\textit{uncertainty}\\ \textit{type B}}}$$

where

$$\mathrm{Var}_0(\hat\beta) =$$
$$= (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}.$$

Proof is elementary. It is enough to determine $\mathrm{Var}_0(\hat\beta) \equiv \mathrm{Var}_0(\hat\beta)|_{\mathbf{\Sigma}_{\hat\Theta}=0}$ and $\mathrm{Var}(\hat\beta)$.

$$\begin{aligned}\mathrm{Var}(\hat\beta) = {} & \mathrm{Var}_0(\hat\beta) + \mathrm{Var}\{\langle\{\mathbf{I} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}'(\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}')^{-1}\mathbf{B}\} \\ & \times (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{D} - (\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}' \\ & \times [\mathbf{B}(\mathbf{X}_2'\mathbf{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{B}']^{-1}\mathbf{C}\rangle\widehat{\Theta}\}.\end{aligned}$$

□

Corollary 2.3 *The standard estimator for the case of the model, where* $\mathbf{X}_2 = \mathbf{I}$ *and* $\mathbf{D} = \mathbf{0}$ *is*

$$\hat\beta = [\mathbf{I} - \mathbf{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}](\mathbf{Y}_2 - \mathbf{D}\widehat{\Theta}) - \mathbf{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}(\mathbf{C}\widehat{\Theta} + \mathbf{a})].$$

Corollary 2.4 *The covariance matrix of the standard estimator for the case of the model, where* $\mathbf{X}_2 = \mathbf{I}$ *and* $\mathbf{D} = \mathbf{0}$*, is*

$$\begin{aligned}\mathrm{Var}(\hat\beta) = {} & [\mathbf{I} - \mathbf{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}]\mathbf{\Sigma}_{2,2}[\mathbf{I} - \mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{2,2}] \\ & + \mathbf{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{C}\,\mathrm{Var}(\hat\Theta)\mathbf{C}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{2,2},\end{aligned}$$

or equivalently

$$\begin{aligned}\mathrm{Var}(\hat\beta) = {} & \mathbf{\Sigma}_{2,2} - \mathbf{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{2,2} + \mathbf{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{C}\mathbf{\Sigma}_{1,1} \\ & \times \mathbf{C}'(\mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}\mathbf{\Sigma}_{2,2}.\end{aligned}$$

Definition 2.5 Let $\mathbf{H}$ be a given $k_2 \times k_2$ positive semidefinite matrix. The estimator $\widetilde{\widetilde{\beta}}$ from the class $\widetilde{\mathcal{U}}_\beta$ is $\mathbf{H}$-optimal if it minimizes the function

$$\phi(\widetilde{\widetilde{\beta}}) = Tr[\mathbf{H}\mathrm{Var}(\widetilde{\widetilde{\beta}})], \quad \widetilde{\widetilde{\beta}} \in \widetilde{\mathcal{U}}_\beta.$$

Theorem 2.4 *If the estimator* $\widetilde{\widetilde{\beta}}$ *from the class* $\widetilde{\mathcal{U}}_\beta$ *is* $\mathbf{H}$*-optimal, then matrices* $\mathbf{X}_2^-, \mathbf{B}^-, \mathbf{W}_1, \mathbf{W}_2$ *(* $^-$ *means g-inverse) in Lemma 2.2 are solutions of the following equation*

$$\mathbf{U}_1\left(\mathbf{W}_1,\ \mathbf{W}_2\right)\begin{pmatrix}\mathbf{V}_1, & \mathbf{T}_1\\ \mathbf{V}_2, & \mathbf{T}_2\end{pmatrix} = \left(\mathbf{P}_1,\ \mathbf{P}_2\right),$$

where

$$\mathbf{U}_1 = [\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}],$$

$$\mathbf{V}_1 = (\mathbf{I} - \mathbf{X}_2\mathbf{X}_2^-)[\boldsymbol{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'](\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'),$$

$$\begin{aligned}\mathbf{V}_2 = {} & \mathbf{B}\mathbf{X}_2^-[\boldsymbol{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'][\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'] \\ & - \mathbf{C}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'[\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2',\end{aligned}$$

$$\begin{aligned}\mathbf{P}_1 = {} & -[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-[\boldsymbol{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'] \\ & \times [\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'] - [\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}\mathbf{B}^-\mathbf{C}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'][\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2'],\end{aligned}$$

$$\begin{aligned}\mathbf{T}_1 = {} & [\mathbf{I} - (\mathbf{X}_2^-)'\mathbf{X}_2']\{[\boldsymbol{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'] \\ & \times (\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'\},\end{aligned}$$

$$\begin{aligned}\mathbf{T}_2 = {} & \mathbf{B}\mathbf{X}_2^-[\boldsymbol{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'](\mathbf{X}_2^-)'\mathbf{B}' + \mathbf{C}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' \\ & - \mathbf{C}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'(\mathbf{X}_2^-)'\mathbf{B}' - \mathbf{B}\mathbf{X}_2^-\mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}',\end{aligned}$$

$$\begin{aligned}\mathbf{P}_2 = {} & -[\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-[\boldsymbol{\Sigma}_{2,2} + \mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'](\mathbf{X}_2^-)'\mathbf{B}' \\ & + [\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}\mathbf{B}^-\mathbf{C}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}' \\ & - [\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}\mathbf{B}^-\mathbf{C}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}'(\mathbf{X}_2^-)'\mathbf{B}' \\ & + [\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)']\mathbf{H}[\mathbf{I} - \mathbf{B}^-\mathbf{B}]\mathbf{X}_2^-\mathbf{D}(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{C}'.\end{aligned}$$

Proof [1], p. 653.

3 Numerical studies—constraints type I

In this part we will concentrate on a numerical calculation of the estimator of parameters. In all following examples we need to construct a condition expressing a relation between parameters of the first and the second stages. From this condition we can always construct a vector function $\mathbf{g}$ of parameter β and Θ where $\mathbf{g}(\beta, \Theta) = \mathbf{0}$. We apply the Taylor expansion at point (β_0, Θ_0) to this function. So for estimators of parameters we get the condition

$$\mathbf{g}(\hat{\beta}, \widehat{\Theta}) = \mathbf{g}(\beta_0, \Theta_0) + \mathbf{C}\delta\widehat{\Theta} + \mathbf{B}\delta\hat{\beta} = \mathbf{0}.$$

We could not change the value $\widehat{\Theta}$ in connecting measurements, and so we must consider

$$\mathbf{g}(\hat{\beta}, \widehat{\Theta}) = \mathbf{g}(\beta_0, \widehat{\Theta}) + \mathbf{B}\delta\hat{\beta} = \mathbf{0}.$$

On basis of these accounts we get the statement

$$\delta\hat{\beta} = [\mathbf{I} - \boldsymbol{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\boldsymbol{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{B}](\mathbf{Y}_2 - \beta_0) - \boldsymbol{\Sigma}_{2,2}\mathbf{B}'(\mathbf{B}\boldsymbol{\Sigma}_{2,2}\mathbf{B}')^{-1}\mathbf{g}(\beta_0, \widehat{\Theta}).$$

Example 3.1 Let us have the elevations Θ_1 and Θ_2 of points A and B, their values were estimated by values $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$. The problem is how to find the elevation of inner point P (see. Figure 1) by means of measured values Y_1 and Y_2 of elevations β_1 and β_2 between points A and P and between points P and B. The accuracy of estimated values $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$ is characterized by standard deviations; eventually can be determined by covariance matrix (below) and analogously it is valid for random variables Y_1 and Y_2 which characterize the measurement of the parameters β_1 and β_2.

Let Θ_1, Θ_2 be parameters of the first stage (connecting) and β_1, β_2 be parameters of the second stage (connected). The estimations $\widehat{\Theta}_1, \widehat{\Theta}_2$ of differences Θ_1, Θ_2 are given from the first stage, the measurement of values Y_1, Y_2 parameters β_1, β_2 are done in the second stage of measurements.

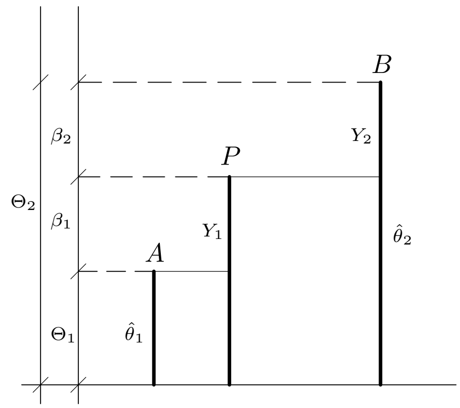

Figure 1: Model of estimation height of inner point

Let us find estimators for the values $(\widehat{\Theta}_1, \widehat{\Theta}_2) = (150, 400.1)$ and $(Y_1, Y_2) = (125, 125)$.[1] Values of variables $\Theta_1, \Theta_2, \beta_1$ and β_2, etc. are indicated in meters.

Values of covariance matrices are indicated in m^2 (for example $\sqrt{\sigma_1^2} = 0.04\,\mathrm{m}$).

We construct a model of connecting measurements in Definition 2.1.

Let $\widehat{\Theta}_1, \widehat{\Theta}_2$ be random variables with mean values Θ_1, Θ_2 and with dispersions τ_1^2, τ_2^2,

$$\mathbf{Y}_1 = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \end{pmatrix} \sim N_2 \left[X_1 \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}; \Sigma_{11} \right].$$

In our case we will consider[2]

$$\Sigma_{11} = \begin{pmatrix} \tau_1^2, & 0 \\ 0, & \tau_2^2 \end{pmatrix} = \begin{pmatrix} 0.0009, & 0.0002, \\ 0.0002, & 0.0007, \end{pmatrix}, \qquad X_1 = I_{2,2}.$$

Let Y_1, Y_2 be stochastically independent random variables with mean values β_1, β_2 and with dispersions σ_1^2, σ_2^2,

$$\mathbf{Y}_2 = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N_2 \left[X_2 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}; \Sigma_{22} \right].$$

In our case we will consider

$$\Sigma_{22} = \begin{pmatrix} \sigma_1^2, & 0 \\ 0, & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 0.0016, & 0.0000, \\ 0.0000, & 0.0016, \end{pmatrix}, \qquad X_2 = I_{2,2}.$$

[1] When we admit that $\widehat{\Theta}_1$, Y_1 and Y_2 are exact values, then it should be $\widehat{\Theta}_2 = 400$ m.

[2] Assumption $X_1 = I_{2,2}$ means that values Θ_1, Θ_2 are measured directly.

One can observe in Figure 1 the following condition is implied for parameters of I. stage Θ_1, Θ_2 and parameters of II. stage β_1, β_2:

$$\beta_1 + \beta_2 = \Theta_2 - \Theta_1. \qquad (c1)$$

In our case we can write the estimator from the class $\widetilde{\mathcal{U}}_\beta$ (see Lemma 2.2) in this form:

$$\widetilde{\widetilde{\beta}} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} k \\ -1-k \end{pmatrix} (Y_1 + Y_2 + \widehat{\Theta}_2 - \widehat{\Theta}_1).$$

Thus, we have for the covariance matrix:

$$\mathrm{Var}(\widetilde{\widetilde{\beta}}) = \begin{pmatrix} s_{11},\ s_{12} \\ s_{21},\ s_{22} \end{pmatrix},$$

where

$$\begin{aligned} s_{11} &= \quad k^2(\tau_1^2 + \tau_2^2 + \sigma_2^2) + (1+k)^2\sigma_1^2, \\ s_{12} &= -k(1+k)(\tau_1^2 + \tau_2^2) - (1+k)^2(\sigma_1^2 - k^2\sigma_2^2), \\ s_{21} &= -k(1+k)(\tau_1^2 + \tau_2^2) - (1+k)^2(\sigma_1^2 - k^2\sigma_2^2), \\ s_{22} &= \quad (1+k)^2(\tau_1^2 + \tau_2^2 + \sigma_1^2) + k^2\sigma_2. \end{aligned}$$

As we can see, it is impossible to find any jointly efficient estimator. Now we will determine numerically the standard estimator $\hat{\beta}$ (see Corollary 2.3), and its covariance matrix $\mathrm{Var}(\hat{\beta})$ (see Corollary 2.4).

At first we will construct the function $g(\beta, \Theta) = \beta_1 + \beta_2 + \Theta_1 - \Theta_2$ from our condition (c1). We will use the Taylor expansion at point (β^0, Θ^0) for this function in the form

$$(B_1, B_2)\delta\beta + (C_1, C_2)\delta\Theta + a = 0,$$

where

$$B_1 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \beta_1} = 1, \qquad B_2 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \beta_2} = 1,$$

$$C_1 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \Theta_1} = 1, \qquad C_2 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \Theta_2} = -1,$$

$$a = g(\beta^0, \Theta^0) = \left(\beta_1^0 + \beta_2^0 + \theta_1^0 - \theta_2^0\right).$$

From approximate values $\Theta_1^0 = 150.0, \Theta_2^0 = 400.1, \beta_1^0 = 125, \beta_2^0 = 125$ we will determine $a = 150.0 - 400.1 + 125.0 + 125.0 = -0.1$.

In our linearized model we will determine from Corollary 2.3 and Corollary 2.4:

$$\hat{\beta} = \begin{pmatrix} 125.05 \\ 125.05 \end{pmatrix}, \qquad \mathrm{Var}(\hat{\beta}) = \begin{pmatrix} 1.1 \cdot 10^{-3} & -5.0 \cdot 10^{-4} \\ -5.0 \cdot 10^{-4} & 1.1 \cdot 10^{-3} \end{pmatrix}.$$

Furthermore we will determine numerically **H**-optimum estimator $\tilde{\tilde{\beta}}$ for the matrix

$$\mathbf{H}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

according to the relationship in Lemma 2.2. We determine matrices $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$ and $\mathbf{W}_2$ from Theorem 2.4; and its covariance matrix $\mathrm{Var}(\tilde{\tilde{\beta}})$ from the relationship from Corollary 2.2

$$\tilde{\tilde{\beta}} = \begin{pmatrix} 125.05 \\ 125.05 \end{pmatrix}, \qquad \mathrm{Var}(\tilde{\tilde{\beta}}) = \begin{pmatrix} 1.1 \cdot 10^{-3} & -5.0 \cdot 10^{-4} \\ -5.0 \cdot 10^{-4} & 1.1 \cdot 10^{-3} \end{pmatrix}.$$

As we can see, the estimator $\tilde{\tilde{\beta}}$ is the same as the estimator $\hat{\beta}$. The estimated elevation of the point P is $\widehat{\Theta}_1 + \hat{\beta}_1 = \widehat{\Theta}_1 + \tilde{\tilde{\beta}}_1 = 150\,\mathrm{m} + 125.05 = 275.05$.

In this case the estimator $\tilde{\tilde{\beta}}$, which we got for chosen matrix **H** is the same as the standard estimator $\hat{\beta}$. Our aim was to show, that it can occur the situation we cannot find any better estimation than the standard estimation. In other examples we show, that generally we can find better estimator. Furthermore our aim was to show using Taylor's expansion, which is used in almost all non-linear situations, according to our aspiration to demonstrate the universal approach for numerical solutions.

Example 3.2 Let us have A and B points with their elevations Θ_1 and Θ_2 measured in the first stage by the values $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$. The problem is to estimate as exactly as possible the elevation β_1 at the inner point P_1 by means of measured values Y_1, Y_2 and Y_3 (see Figure 2).

The accuracy in determination of the values $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$ of heights Θ_1 and Θ_2 is characterized by the standard deviations, or by the covariance matrix (see follow up) and analogously of measured values Y_1, Y_2 and Y_3 of the values β_1, β_2 and β_3.

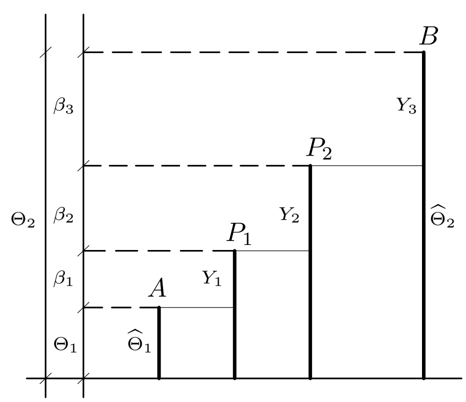

Figure 2: Model with two inner points

Now let us determine the standard estimator and the **H**-optimum estimator and their covariance matrices for the values $(\widehat{\Theta}_1, \widehat{\Theta}_2) = (125.00, 575.09)$ and $(Y_1, Y_2, Y_3) = (100.00, 150.00, 200.00)$.[3]

We will construct a model of connecting measurement according to Definition 2.1.

Let $\widehat{\Theta}_1, \widehat{\Theta}_2$ be random variables with mean values Θ_1, Θ_2 and with the dispersions τ_1^2, τ_2^2,

$$\mathbf{Y}_1 = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \end{pmatrix} \sim N_2 \left[X_1 \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}; \Sigma_{11} \right].$$

In our case we will consider

$$\Sigma_{11} = \begin{pmatrix} \tau_1^2, & 0 \\ 0, & \tau_2^2 \end{pmatrix} = \begin{pmatrix} 0.0009, & 0.0002, \\ 0.0002, & 0.0007, \end{pmatrix}, \qquad X_1 = I_{2,2}.$$

Let Y_1, Y_2, Y_3 be stochastically independent random variables with mean values $\beta_1, \beta_2, \beta_3$ and with the dispersions $\sigma_1^2, \sigma_2^2, \sigma_3^2$,

$$\mathbf{Y}_2 = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3 \left[X_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}; \Sigma_{22} \right].$$

In our case we will consider

$$\Sigma_{22} = \begin{pmatrix} \sigma_1^2, & 0 & 0 \\ 0, & \sigma_2^2 & 0 \\ 0, & 0, & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 0.0016, & 0.0000 & 0.0000, \\ 0.0000, & 0.0016 & 0.0000, \\ 0.0000, & 0.0000 & 0.0016, \end{pmatrix}, \qquad X_2 = I_{2,2}.$$

One can observe in Figure 2 that the following condition is implied for parameters of the first stage Θ_1, Θ_2 and parameters of the second stage β_1, β_2 and β_3:

$$\beta_1 + \beta_2 + \beta_3 = \Theta_2 - \Theta_1. \qquad (c2)$$

We will calculate numerically a standard estimator $\hat{\beta}$ and **H**-optimum estimator $\tilde{\tilde{\beta}}$ like in previous example.

First of all we will construct the function $g(\beta, \Theta) = \beta_1 + \beta_2 + \beta_3 + \Theta_1 - \Theta_2$ from our condition (c2). We will construct the Taylor expansion at point (β^0, Θ^0) in the form

$$(B_1, B_2, B_3)\delta\beta + (C_1, C_2)\delta\Theta + a = 0,$$

where

$$B_1 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \beta_1} = 1, \quad B_2 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \beta_2} = 1, \quad B_3 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \beta_3} = 1,$$

$$C_1 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \Theta_1} = 1, \quad C_2 = \frac{\partial g(\beta^0, \Theta^0)}{\partial \Theta_2} = -1,$$

[3] If we admitted that the values $\widehat{\Theta}_1$, Y_1, Y_2 and Y_3 are exact values, then it must be $\widehat{\Theta}_2 = 575.00$ m.

$$a = \left(\beta_1^0 + \beta_2^0 + \beta_3^0 + \theta_1^0 - \theta_2^0\right).$$

From the approximate values $\Theta_1^0 = 125.00$, $\Theta_2^0 = 575.09$, $\beta_1^0 = 100.00$, $\beta_2^0 = 150.00$, $\beta_3^0 = 200.00$ we receive $a = 100.00 + 150.00 + 200.00 + 125.00 - 575.09 = -0.09$.

In our linearized model we will numerically determine the estimator and the covariance matrix from the Corollary 2.3 and the Corollary 2.4:

$$\hat{\beta} = \begin{pmatrix} 100.030 \\ 150.030 \\ 200.030 \end{pmatrix}, \qquad \mathrm{Var}(\hat{\beta}) = \begin{pmatrix} 1.2\cdot10^{-3} & -4.0\cdot10^{-4} & -4.0\cdot10^{-4} \\ -4.0\cdot10^{-4} & 1.2\cdot10^{-3} & -4.0\cdot10^{-4} \\ -4.0\cdot10^{-4} & -4.0\cdot10^{-4} & 1.2\cdot10^{-3} \end{pmatrix}.$$

After that we will numerically calculate the $\mathbf{H}$-optimum estimator $\hat{\beta}$ for the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

according to Lemma 2.2 and its covariance matrix according to Corollary 2.2. The matrices $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$ and $\mathbf{W}_2$ we determine from the Theorem 2.4

$$\widetilde{\widetilde{\beta}} = \begin{pmatrix} 100.024 \\ 150.033 \\ 200.033 \end{pmatrix}, \qquad \mathrm{Var}(\widetilde{\widetilde{\beta}}) = \begin{pmatrix} 1.173\cdot10^{-3} & -4.267\cdot10^{-4} & -4.267\cdot10^{-4} \\ -4.167\cdot10^{-4} & 1.233\cdot10^{-3} & -3.667\cdot10^{-4} \\ -4.267\cdot10^{-4} & -3.667\cdot10^{-4} & 1.233\cdot10^{-3} \end{pmatrix}.$$

Next we will calculate $\mathrm{Tr}(\mathbf{H}\,\mathrm{Var}(\widetilde{\widetilde{\beta}})) = 1.173\cdot10^{-3}$.

These estimators $\hat{\beta}$ and $\widetilde{\widetilde{\beta}}$ are typically different in this case.

The elevation between the points A and P_1 obtained by the standard estimator is $\hat{\beta}_1 = 150.030$.

The elevation between the points A and P_1 obtained by the $\mathbf{H}$-optimum estimator is $\widetilde{\widetilde{\beta}}_1 = 150.024$.

By choosing the matrix $\mathbf{H}$ which minimized a dispersion in estimator of the first component of the vector $\widetilde{\widetilde{\beta}}$ we got better estimator for the elevation between the points A and P_1 in comparison with the standard estimator $\hat{\beta}$. This follows from the fact, that for the chosen matrix $\mathbf{H}$ it is $\mathrm{Tr}(\mathbf{H}\,\mathrm{Var}(\widetilde{\widetilde{\beta}})) = 1.173\cdot10^{-3} < 1.200\cdot10^{-3} = \mathrm{Var}_{11}(\hat{\beta})$.

Example 3.3 The aim is to find an estimator for the plane coordinates of the points P_1 and P_2 in a cartesian co-ordinates from the Figure 3. We have the measured values $\widehat{\Theta}_1, \widehat{\Theta}_2$ of coordinates Θ_1, Θ_2 of the point A, the measured values $\widehat{\Theta}_3, \widehat{\Theta}_4$ of coordinates Θ_3, Θ_4 of the point B, the measured values Y_1, Y_2, Y_3 of lengths β_1, β_2 and β_3 and the measured values Y_4, Y_5 of angles β_4 and β_5 (see Figure 3).

Let $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ be parameters of the first stage (connecting) and β_1, β_2, β_3, β_4, β_5 be parameters of the second stage (connected). The aim of the measurements is to determine the values $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\beta}_3$, $\hat{\beta}_4$, $\hat{\beta}_5$, when the estimators $\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, \widehat{\Theta}_4$ of the coordinates $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ are given from the first stage of measurements. The measurements Y_1, Y_2, Y_3, Y_4, Y_5 of the parameters $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are done in the second stage of measurements.

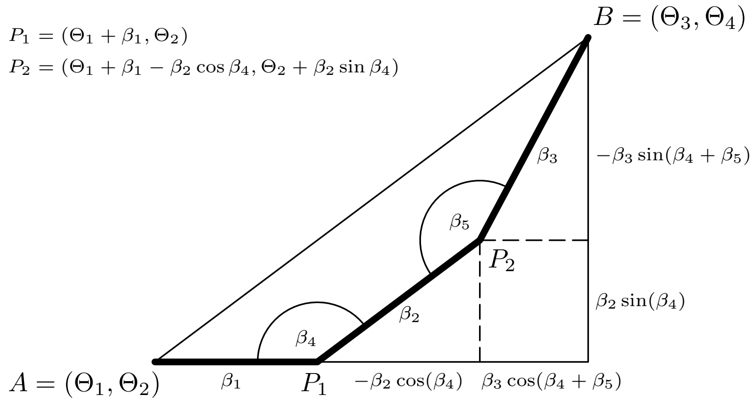

Figure 3: Model for determining distance on encastered polygon

In our model we will determine estimators and their covariance matrices for the result of measurements $(\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, \widehat{\Theta}_4) = (0, 0, 640.1, 480.1)$ and the result $(Y_1, Y_2, Y_3, Y_4, Y_5) = (240, 300, 340, 2.498091546, 2.70425476)$.

The values $\widehat{\Theta}_1, \widehat{\Theta}_2, \widehat{\Theta}_3, \widehat{\Theta}_4$, Y_1, Y_2, Y_3, etc. are in meters. The values of the angles Y_4, Y_5 are written in radians.

The accuracy of measurements was given by the covariance matrices. Let $\widehat{\Theta}_1$, $\widehat{\Theta}_2$, $\widehat{\Theta}_3$, $\widehat{\Theta}_4$ be random variables with mean values Θ_1, Θ_2, Θ_3, Θ_4,

$$\mathbf{Y}_1 = \begin{pmatrix} \widehat{\Theta}_1 \\ \widehat{\Theta}_2 \\ \widehat{\Theta}_3 \\ \widehat{\Theta}_4 \end{pmatrix} \sim N_4 \left[\mathbf{X}_1 \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{pmatrix} ; \mathbf{\Sigma}_1 \right].$$

In our case we will consider

$$\mathbf{\Sigma}_{1,1} = \begin{pmatrix} 0,0016, & 0,0002, & 0,0004, & 0,0000 \\ 0,0002, & 0,0016, & 0,0002, & 0,0000 \\ 0,0004, & 0,0002, & 0,0016, & 0,0005 \\ 0,0000, & 0,0000, & 0,0005, & 0,0016 \end{pmatrix}, \qquad \mathbf{X}_1 = \mathbf{I}_{4,4}.$$

Let Y_1, Y_2, Y_3, Y_4, Y_5 be stochastically independent random variables with mean values $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ and with dispersion $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2, \sigma_5^2$,

$$\mathbf{Y}_2 = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} \sim N_5 \left[\mathbf{X}_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} ; \mathbf{\Sigma}_2 \right].$$

In our case we will consider

$$\mathbf{\Sigma}_{2,2} = \begin{pmatrix} 0.0016, & 0.0000, & 0.0000, & 0.0000 & 0.0000 \\ 0.0000, & 0.0016, & 0.0000, & 0.0000 & 0.0000 \\ 0.0000, & 0.0000, & 0.0016, & 0.0000 & 0.0000 \\ 0.0000, & 0.0000, & 0.0000, & (\frac{10}{206265})^2, & 0.0000 \\ 0.0000, & 0.0000, & 0.0000, & 0.0000, & (\frac{10}{206265})^2 \end{pmatrix}, \qquad \mathbf{X}_2 = \mathbf{I}_{5,5}.$$

One can observe in Figure 3 the following condition is implied for the parameters of the first stage $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ and for the parameters of the second stage $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$:

$$(\Theta_3 - \Theta_1)^2 + (\Theta_4 - \Theta_2)^2 = x^2 + y^2, \qquad (c3)$$

where

$$\begin{aligned} x &= \beta_1 - \beta_2 \cos(\beta_4) + \beta_3 \cos(\beta_4 + \beta_5) \\ y &= \beta_2 \sin(\beta_4) - \beta_3 \sin(\beta_4 + \beta_5). \end{aligned}$$

As in the previous examples we will calculate numerically the standard estimator $\hat{\beta}$ and the **H**-optimum estimator $\tilde{\tilde{\beta}}$.

First of all we will construct the following function from our condition (c3):

$$\begin{aligned} \mathbf{g}(\beta, \Theta) = {} & (\Theta_3 - \Theta_1)^2 + (\Theta_4 - \Theta_2)^2 - (\beta_1^2 - 2\beta_1\beta_2 \cos(\beta_4) + \beta_2^2 \\ & + 2\beta_1\beta_3 \cos(\beta_4 + \beta_5) - 2\beta_2\beta_3 \cos(\beta_4) \cos(\beta_4 + \beta_5) + \\ & + \beta_3^2 - 2\beta_2\beta_3 \sin(\beta_4) \sin(\beta_4 + \beta_5)). \end{aligned}$$

We will generate the Taylor expansion at point (β^0, Θ^0) for the above function in the form

$$(B_1, B_2, B_3, B_4, B_5)\delta\beta + (C_1, C_2, C_3, C_4)\delta\Theta + a = 0$$

where $B_1 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \beta_1}$, $B_2 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \beta_2}$, $B_3 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \beta_3}$, $B_4 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \beta_4}$, $B_5 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \beta_5}$, $C_1 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \Theta_1}$, $C_2 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \Theta_2}$, $C_3 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \Theta_3}$, $C_4 = \frac{\partial g(\beta^0,\Theta^0)}{\partial \Theta_4}$, $a = g(\beta^0, \Theta^0)$.

We will determine the appropriate partial derivative and determine the value a

$$\begin{aligned} B_1 &= -2\beta_{1,0} + 2\beta_{2,0} \cos(\beta_{4,0}) - 2\beta_{3,0} \cos(\beta_{4,0} + \beta_{5,0}), \\ B_2 &= 2\beta_{1,0} \cos(\beta_{4,0}) - 2\beta_{2,0} + 2\beta_{3,0} \cos(\beta_{4,0}) \cos(\beta_{4,0} + \beta_{5,0}) + 2\beta_{3,0} \sin(\beta_{4,0}) \\ &\quad \times \sin(\beta_{4,0} + \beta_{5,0}), \\ B_3 &= -2\beta_{1,0} \cos(\beta_{4,0} + \beta_{5,0}) + 2\beta_{2,0} \cos(\beta_{4,0}) \cos(\beta_{4,0} + \beta_{5,0}) \\ &\quad - 2(\beta_{3,0}) + 2\beta_{2,0} \sin(\beta_{4,0}) \sin(\beta_{4,0} + \beta_{5,0}), \end{aligned}$$

$$
\begin{aligned}
B_4 &= -2\beta_{1,0}\beta_{2,0}\sin(\beta_{4,0}) + 2\beta_{1,0}\beta_{3,0}\sin(\beta_{4,0}+\beta_{5,0}) + 2\beta_{2,0}\beta_{3,0} \\
&\quad \times \left(-\sin(\beta_{4,0})\cos(\beta_{4,0}+\beta_{5,0}) - \cos(\beta_{4,0})\sin(\beta_{4,0}+\beta_{5,0})\right) \\
&\quad + 2\beta_{2,0}\beta_{3,0}\left(\cos(\beta_{4,0})\sin(\beta_{4,0}+\beta_{5,0}) + \sin(\beta_{4,0})\cos(\beta_{4,0}+\beta_{5,0})\right), \\
B_5 &= -2\beta_{1,0}\beta_{3,0}\sin(\beta_{4,0}+\beta_{5,0}) + 2\beta_{2,0}\beta_{3,0}\cos(\beta_{4,0}) \\
&\quad \times \sin(\beta_{4,0}+\beta_{5,0}) - 2\beta_{2,0}\beta_{3,0}\cos(\beta_{4,0}+\beta_{5,0}), \\
C_1 &= -2(\theta_{3,0}-\theta_{1,0}), \quad C_2 = -2(\theta_{4,0}-\theta_{2,0}), \\
C_3 &= 2(\theta_{3,0}-\theta_{1,0}), \quad C_4 = 2(\theta_{4,0}-\theta_{2,0}), \\
a &= (\theta_{3,0}-\theta_{1,0})^2 + (\theta_{4,0}-\theta_{2,0})^2 - \beta_{1,0}^2 + 2\beta_{1,0}\beta_{2,0}\cos(\beta_{4,0}) - \beta_{2,0}^2 \\
&\quad - 2\beta_{1,0}\beta_{3,0}\cos(\beta_{4,0}+\beta_{5,0}) + 2\beta_{2,0}\beta_{3,0}\cos(\beta_{4,0})\cos(\beta_{4,0}+\beta_{5,0}) - \beta_{3,0}^2 \\
&\quad + 2\beta_{2,0}\beta_{3,0}\sin(\beta_{4,0})\sin(\beta_{4,0}+\beta_{5,0}).
\end{aligned}
$$

By choosing

$$\beta_0 = (\beta_{1,0}, \beta_{2,0}, \beta_{3,0}, \beta_{4,0}, \beta_{5,0}) = (240, 300, 340, 2.498091546, 2.70425476)$$

and $\Theta_0 = (\Theta_{1,0}, \Theta_{2,0}, \Theta_{3,0}, \Theta_{4,0})$ we get $B_1 = -1280$, $B_2 = -1600$, $B_3 = -1449$, $B_4 = -230400$, $B_5 = -230400$, $C_1 = -1280$, $C_2 = -960$, $C_3 = 1280$, $C_4 = 960$, $a = -224.02$.

In our linearized model we will determine numerically the estimator and the covariance matrix from the Corollary 2.3 and the Corollary 2.4:

$$\hat{\beta} = \begin{pmatrix} 240.044 \\ 300.056 \\ 340.050 \\ 2.49810329204 \\ 2.70426650604 \end{pmatrix},$$

$$\mathrm{Var}(\hat{\beta}) = \begin{pmatrix}
1.5129\cdot 10^{-3} & -1.0888\cdot 10^{-4} & -9.8631\cdot 10^{-5} & -2.3032\cdot 10^{-8} & -2.3032\cdot 10^{-8} \\
-1.0888\cdot 10^{-4} & 1.4639\cdot 10^{-3} & -1.2329\cdot 10^{-4} & -2.8790\cdot 10^{-8} & -2.8790\cdot 10^{-8} \\
-9.8631\cdot 10^{-5} & -1.2329\cdot 10^{-4} & 1.4883\cdot 10^{-3} & -2.6080\cdot 10^{-8} & -2.6080\cdot 10^{-8} \\
-2.3032\cdot 10^{-8} & -2.8790\cdot 10^{-8} & -2.6080\cdot 10^{-8} & 2.3443\cdot 10^{-9} & -6.0903\cdot 10^{-12} \\
-2.3032\cdot 10^{-8} & -2.8790\cdot 10^{-8} & -2.6080\cdot 10^{-8} & -6.0903\cdot 10^{-12} & 2.3443\cdot 10^{-9}
\end{pmatrix}.$$

After that we will numerically determine the $\mathbf{H}$-optimum estimator $\hat{\beta}$ for the matrix

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

according to Lemma 2.2 and its covariance matrix according to Corollary 2.2. We determine matrices $\mathbf{X}_2^-$, $\mathbf{B}^-$, $\mathbf{W}_1$ and $\mathbf{W}_2$ according to the Theorem 2.4 in this way:

$$\begin{aligned}
\mathbf{U}_1 &= -(\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)')\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B}),\\
\mathbf{V}_1 &= \mathbf{0}, \quad \mathbf{V}_2 = \mathbf{0}, \quad \mathbf{T}_1 = \mathbf{0},\\
\mathbf{P}_1 &= -(\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)')\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{\Sigma}_{2,2} = \mathbf{0},\\
\mathbf{T}_2 &= \mathbf{B}\mathbf{\Sigma}_{2,2}\mathbf{B}' + \mathbf{C}\mathbf{\Sigma}_{1,1}\mathbf{C}',\\
\mathbf{P}_2 &= -(\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)')\mathbf{H}(\mathbf{I} - \mathbf{B}^-\mathbf{B})\mathbf{\Sigma}_{2,2}\mathbf{B}' + (\mathbf{I} - \mathbf{B}'(\mathbf{B}^-)')\mathbf{H}\mathbf{B}^-\mathbf{C}\mathbf{\Sigma}_{1,1}\mathbf{C}'.
\end{aligned}$$

Then the matrices $\mathbf{W}_1, \mathbf{W}_2$ are solution of the equations

$$\mathbf{U}_1(\mathbf{W}_1, \mathbf{W}_2)\begin{pmatrix} \mathbf{0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{T}_2 \end{pmatrix} = (\mathbf{0}, \mathbf{P}_2)$$

and we get

$$\mathbf{U}_1(\mathbf{0}, \mathbf{W}_2 T) = (\mathbf{0}, \mathbf{P}_2) \Rightarrow \mathbf{U}_1\mathbf{W}_2\mathbf{T} = \mathbf{P}_2 \Rightarrow \mathbf{W}_2 = \mathbf{U}_1^-\mathbf{P}_2\mathbf{T}_2^-$$

In our case we get from Lemma 2.2:

$$\tilde{\tilde{\beta}} = \begin{pmatrix} 240.025\\ 300.031\\ 340.028\\ 2.49810329204\\ 2.70426650604 \end{pmatrix},$$

$$\operatorname{Var}(\tilde{\tilde{\beta}}) = \begin{pmatrix}
1.3726\cdot 10^{-3} & -2.8430\cdot 10^{-4} & -2.5754\cdot 10^{-4} & -6.0048\cdot 10^{-8} & -6.0048\cdot 10^{-8}\\
-2.8430\cdot 10^{-4} & 1.2446\cdot 10^{-3} & -3.2193\cdot 10^{-4} & -7.5060\cdot 10^{-8} & -7.5060\cdot 10^{-8}\\
-2.5754\cdot 10^{-4} & -3.2193\cdot 10^{-4} & 1.3084\cdot 10^{-3} & -6.7995\cdot 10^{-8} & -6.7995\cdot 10^{-8}\\
-6.0048\cdot 10^{-8} & -7.5060\cdot 10^{-8} & -6.7995\cdot 10^{-8} & 1.9142\cdot 10^{-8} & 1.6791\cdot 10^{-8}\\
-6.0048\cdot 10^{-8} & -7.5060\cdot 10^{-8} & -6.7995\cdot 10^{-8} & 1.6791\cdot 10^{-8} & 1.9142\cdot 10^{-8}
\end{pmatrix}.$$

By chosen matrix $\mathbf{H}$ *minimizing data errors in the process estimation of the vector* $\tilde{\tilde{\beta}}$ *we got better estimator of the parameter* β *in comparison with the standard estimator* $\hat{\beta}$*. It follows from the fact that for the chosen matrix* $\mathbf{H}$ *is* $\operatorname{Tr}(\mathbf{H}\operatorname{Var}(\tilde{\tilde{\beta}})) = 3.9256\cdot 10^{-3} < 4.4651\cdot 10^{-3} = \operatorname{Tr}(\mathbf{H}\operatorname{Var}(\hat{\beta})$.

Let us study the proportion accuracy of the standard estimator $\hat{\beta}_i$ and the $\mathbf{H}_i$-optimum estimator $\tilde{\tilde{\beta}}_i$ for $i = 1, \ldots, 5$. We will not determine the estimators from now, but we will only study the trace of the covariance matrix $\operatorname{Tr}(\mathbf{H}\operatorname{Var}(\tilde{\tilde{\beta}})$ for comparing it with the above mentioned $\operatorname{Tr}(\mathbf{H}\operatorname{Var}(\hat{\beta})$.

For matrix $\mathbf{H}_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$ we get $\begin{array}{l}\operatorname{Tr}(\mathbf{H}_1\operatorname{Var}(\tilde{\tilde{\beta}})) = 1.3726\cdot 10^{-3}\\ < \operatorname{Tr}(\mathbf{H}_1\operatorname{Var}(\hat{\beta})) = 1.5129\cdot 10^{-3},\end{array}$

for matrix $\mathbf{H}_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$ we get $\begin{array}{l}\operatorname{Tr}(\mathbf{H}_2\operatorname{Var}(\tilde{\tilde{\beta}})) = 1.2446\cdot 10^{-3}\\ < \operatorname{Tr}(\mathbf{H}_2\operatorname{Var}(\hat{\beta})) = 1.4639\cdot 10^{-3},\end{array}$

for matrix $\mathbf{H}_3 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}$ we get $\begin{array}{l} \mathrm{Tr}(\mathbf{H}_3 \,\mathrm{Var}(\tilde{\tilde{\beta}})) = 1.3084 \cdot 10^{-3} \\ < \mathrm{Tr}(\mathbf{H}_3 \,\mathrm{Var}(\hat{\beta})) = 1.4883 \cdot 10^{-3}, \end{array}$

for matrix $\mathbf{H}_4 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{pmatrix}$ we get $\begin{array}{l} \mathrm{Tr}(\mathbf{H}_4 \,\mathrm{Var}(\tilde{\tilde{\beta}})) = 2.3345 \cdot 10^{-9} \\ < \mathrm{Tr}(\mathbf{H}_4 \,\mathrm{Var}(\hat{\beta})) = 2.3443 \cdot 10^{-9}, \end{array}$

for matrix $\mathbf{H}_5 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \end{pmatrix}$ we get $\begin{array}{l} \mathrm{Tr}(\mathbf{H}_5 \,\mathrm{Var}(\tilde{\tilde{\beta}})) = 2.3345 \cdot 10^{-9} \\ < \mathrm{Tr}(\mathbf{H}_5 \,\mathrm{Var}(\hat{\beta})) = 2.3443 \cdot 10^{-9}. \end{array}$

It is evident that $\mathrm{Tr}(\mathbf{H}_i \,\mathrm{Var}(\tilde{\tilde{\beta}})) < \mathrm{Tr}(\mathbf{H}_i \,\mathrm{Var}(\hat{\beta}))$ for $i = 1, \dots 5$. Now let us study the proportion of this values for different covariance matrices $\mathbf{\Sigma}_{1,1}$ and $\mathbf{\Sigma}_{2,2}$. In other numerical calculations we choose the matrix $\mathbf{\Sigma}_{1,1}$ as the fixed one and we change the matrix $\mathbf{\Sigma}_{2,2}$ by the multiplication by the number k. The proportions in dependence on k are shown in the following table and graph.

The proportion $\mathrm{Tr}(\mathbf{H}_i \,\mathrm{Var}(\tilde{\tilde{\beta}}))$ and $\mathrm{Tr}(\mathbf{H}_i \,\mathrm{Var}(\hat{\beta}))$					
k	$i = 1, \mathbf{H}_1$	$i = 2, \mathbf{H}_2$	$i = 3, \mathbf{H}_3$	$i = 4, \mathbf{H}_4$	$i = 5, \mathbf{H}_5$
400	100.00 %	100.00 %	100.00 %	100.00 %	100.00%
100	100.00 %	100.00 %	100.00 %	100.00 %	100.00%
64	100.00 %	99.99 %	99.99 %	100.00 %	100.00%
50	99.99 %	99.98 %	99.99 %	100.00 %	100.00%
25	99.97 %	99.94 %	99.96 %	100.00 %	100.00%
16	99.92 %	99.85 %	99.89 %	100.00 %	100.00%
9	99.77 %	99.57 %	99.68 %	99.99 %	99.99%
5	99.31 %	98.73 %	99.04 %	99.97 %	99.97%
4	98.97 %	98.12 %	98.58 %	99.96 %	99.96%
3	98.30 %	96.95 %	97.67 %	99.93 %	99.93%
2	96.68 %	94.21 %	95.51 %	99.87 %	99.87%
1	90.72 %	85.02 %	87.91 %	99.58 %	99.58%
1/2	78.72 %	68.96 %	73.65 %	98.85 %	98.85%
1/4	60.81 %	48.89 %	54.29 %	97.19 %	97.19%
1/10	35.45 %	25.69 %	29.81 %	92.23 %	92.23%
1/16	24.93 %	17.37 %	20.48 %	87.66 %	87.66%
1/25	17.24 %	11.69 %	13.93 %	81.57 %	81.57%
1/50	9.27 %	6.12 %	7.37 %	68.36 %	68.36%
1/64	7.37 %	4.83 %	5.83 %	62.67 %	62.67%
1/100	4.82 %	3.13 %	3.80 %	51.62 %	51.62%
1/400	1.24 %	0.80 %	0.97 %	20.90 %	20.90%

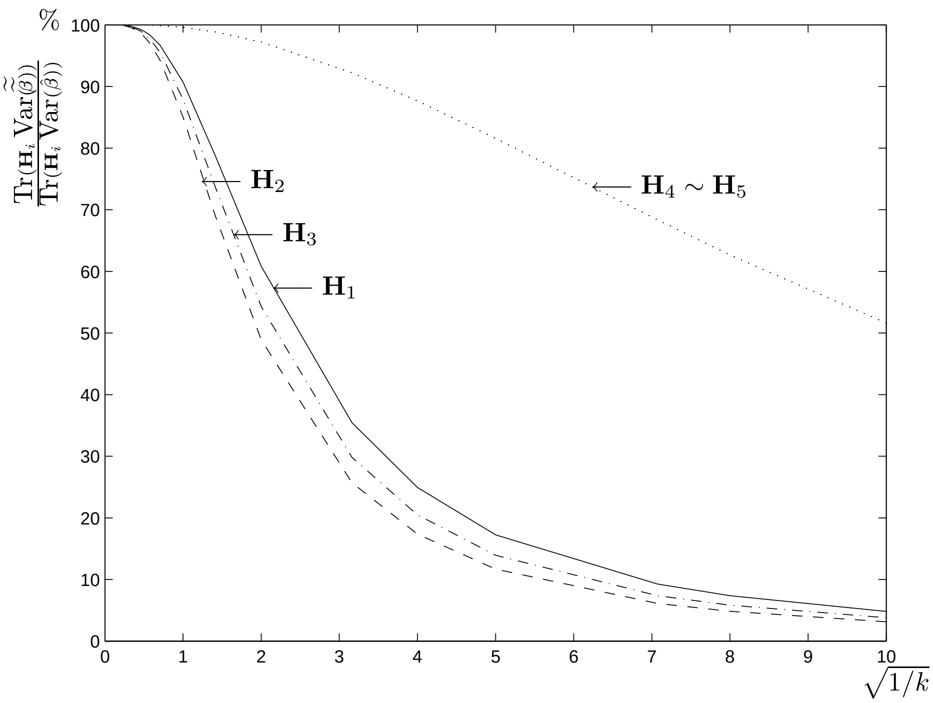

Figure 4: The proportion $\mathrm{Tr}(\mathbf{H}_i\,\mathrm{Var}(\tilde{\tilde{\beta}}))$ and $\mathrm{Tr}(\mathbf{H}_i\,\mathrm{Var}(\hat{\beta}))$

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