Přírodovědecká fakulta
Katedra matematické analýzy a aplikací matematiky

Preprint 12/2004

Jiří Fišer:

Numerical aspects of multivalued fractals

11. 11. 2004

PREPRINT SERIES

NUMERICAL ASPECTS OF MULTIVALUED FRACTALS

JIŘÍ FIŠER

Dept. of Math. Analysis, Fac. of Science, Palacký University,
Tomkova 40, 779 00 Olomouc-Hejčín, Czech Republic
e-mail: fiser@inf.upol.cz

Abstract. At first, we will present the Collage Theorems for iterated multifunction systems and, more generally, for respective continuation principles. Their application will be then discussed in confrontation with numerical (digital) multivalued fractals generated either randomly or (on the basis of an appropriate Shadowing Lemma) in a deterministic way. Some illustrating examples will be given.

Keywords: Iterated multifunction systems, Collage theorems, shadowing, numerics.

AMS Subject Classification: 28A80, 54C60, 65D18.

1. Introduction

We begin with the extension (see [AG01]) of the well-known theorem due to J. E. Hutchinson [Hut81] and M. F. Barnsley [Bar88] based on the well-known Banach contraction principle.

Theorem 1 (cf. [AG01, AFGL]). *Assume that (X,d) is a complete metric space and*

$$\{\varphi_i : X \multimap X,\ i = 1,\ldots,n;\ n \in \mathbb{N}\} \tag{1}$$

is a system of multivalued contractions (with nonempty compact values), i.e.

$$d_H(\varphi_i(x),\varphi_i(y)) \le L_i\, d(x,y), \quad \text{for all}\ \ x,y \in X,\ i = 1,\ldots,n, \tag{2}$$

where $L_i \in [0,1)$, $i = 1,\ldots,n$.

Then there exists exactly one compact invariant subset $A^* \subset X$ of the Hutchinson-Barnsley map

$$F(x) := \bigcup_{i=1}^{n} \varphi_i(x),\ x \in X, \tag{3}$$

called the attractor (fractal) of (1) or, equivalently, exactly one fixed-point $A^* \in \mathcal{K}(X) := \{A \subset X \mid A$ is nonempty and compact$\}$ of the induced Hutchinson-Barnsley operator

$$F^*(A) := \overline{\bigcup_{x\in A} F(x)}\ \left(= \bigcup_{x\in A} F(x)\right),\ A \in \mathcal{K}(X), \tag{4}$$

in the hyperspace $(\mathcal{K}(X), d_H)$, where d_H stands for the Hausdorff metric, defined by

$$d_H(A,B) := \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B)\ \ \text{and}\ \ B \subset O_\varepsilon(A)\},$$

Supported by the Council of Czech Government (J 14/98: 153 100 011).

This paper was presented at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

where $O_\varepsilon(C) = \{x \in X \mid d(x, C) < \varepsilon\}$, *for any nonempty, bounded, closed set* $C \subset X$.

Moreover,

$$\lim_{m\to\infty} d_H(F^{*^m}(A), A^*) = 0, \quad \textit{for every} \quad A \in \mathcal{K}(X), \tag{5}$$

and (Collage)

$$d_H(A, A^*) \le \frac{1}{1-L}\, d_H(A, F^*(A)), \tag{6}$$

where $(1 >) \, L = \max_{i=1,\dots,n} L_i$.

There are several generalizations of this theorem for systems of (weak) contractions, where the notion of weak contractions is understood in different ways (cf. [Pet01, Pet02, PR01, PR]), iterated multifunction systems (IMS) of nonexpansive maps [AFGL] or IMS of compact maps [AF04, Kie02, JGP00, LM00]. Unfortunately, the more generalization or different character is with loss of the constructive part of Theorem 1.

On the other hand, for IMS of weak contractions the uniqueness can be guaranteed as well. Moreover, in [And] J. Andres developed now the continuation principle for IMS of contractions (in the same paper and cf. [AFGL, AG03], continuation principle for compact maps was developed as well). Since the Andres Theorem is based on the Granas continuation technique, which was completed by R. Precup [Pre02] by a computational part, we can simply complete on this base Andres Theorem as follows.

Theorem 2. *Let* $\{\varphi_i : [0,1] \times \overline{S} \multimap X; i = 1, \dots, n\}$ *be a family of multivalued* (L_i, M_i)*-Lipschitz maps with compact values , i.e.* $(i = 1, \dots, n)$

$$d_H(\varphi_i(\lambda, x), \varphi_i(\lambda, y)) \le L_i d(x, y), \ \textit{for all } \lambda \in [0,1] \textit{ and } x, y \in \overline{S}, \tag{7}$$

and

$$\begin{aligned} d_H(\varphi_i(\lambda_1, x), \varphi_i(\lambda_2, x)) &\le M_i|\lambda_1 - \lambda_2|, \ \textit{for some } M_i > 0, \\ &\textit{for all } x \in \overline{S} \textit{ and } \lambda_1, \lambda_2 \in [0,1], \end{aligned} \tag{8}$$

where $L_i \in [0,1)$, $M_i \in (0, \infty)$, *for* $i = 1, \dots, n$, *and* $S \subset X$ *is a subset of a complete metric space* (X, d). *Assume there is* $\overline{\mathcal{U}} \subset \mathcal{K}(\overline{S})$ *such that* $F^*_\lambda \in \mathcal{C}_0(\mathcal{U}, \mathcal{K}(X))$ *(= the set of all contractions* $F^* : \overline{\mathcal{U}} \to \mathcal{K}(X) \mid \operatorname{Fix} F^* \cap \partial\mathcal{U} = \emptyset$*),* $\lambda \in [0,1]$.

In addition suppose that F^*_0 *has a fixed point* $A(0) \in \mathcal{U}$. *Then, for each* $\lambda \in [0,1]$, *there exists a unique fixed point* $A(\lambda) \in \mathcal{U}$ *of* F^*_λ.

Moreover, $A(\lambda)$ *depends continuously on* λ *and there exists* $0 < r \le \infty$, *integers* $m, n_1, n_2, \dots, n_{m-1}$ *and numbers* $0 < \lambda_1 < \lambda_2 < \dots < \lambda_{m-1} < \lambda_m = 1$ *such that for any* $A_0 \in \mathcal{K}(X)$ *satisfying* $d_H(A_0, A(0)) \le r$, *the sequences* $(A_{j,k})_{k\ge 0}$, $j = 1, 2, \dots, m$,

$$\begin{aligned} &A_{1,0} = A_0 \\ &A_{j,k+1} = F^*_{\lambda_j}(A_{j,k}), \quad k = 0, 1, \dots \\ &A_{j+1,0} = A_{j,n_j}, \quad j = 1, 2, \dots, m-1 \end{aligned}$$

are well defined and satisfy

$$d_H(A_{j,k}, A(\lambda_j)) \le \frac{L^k}{1-L} d_H(A_{j,0}, F^*_{\lambda_j}(A_{j,0})) \ \ (L := \max\{L_i \mid i = 1, 2, \dots, n\}, k \in \mathbb{N}).$$

To present the computational part of Theorem 2 in more detail, we suppose the assumptions of the theorem to be satisfied, and a unique fixed point $A(0)$ of F_0^* to exists. We wish to obtain an approximation $\widetilde{A}_1$ of $A(1)$ with $d_H(\widetilde{A}_1, A(1)) \leq \varepsilon$.

We start with some $A_0 \in \mathcal{U}$, which is an r-approximation of $A(0)$ (i.e. $d_H(A_0, A(0)) \leq r$), where

$$r \leq \inf \{ d_H(A(\lambda), B) \mid B \in \partial\mathcal{U}, \lambda \in [0,1] \}.$$

Furthermore, we find $h > 0$ such that for every $A \in \overline{\mathcal{U}}$ and $\lambda, \mu \in [0,1]$, $|\lambda - \mu| \leq h$,

$$d_H(F_\lambda^*(A), F_\mu^*(A)) \leq (1-L)r,$$

which guarantees

$$d_H(A(\lambda), A(\mu)) \leq r.$$

Finally, for $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{m-1} < \lambda_m = 1$ such that $\lambda_{j+1} - \lambda_j \leq h$, $j = 0, 1, \ldots m-1$, we can compute the sequences

$$A_{j,k}, \quad k = 0, 1, 2, \ldots, n_j, \quad j = 1, 2, \ldots, m,$$

in the following way.

For $j = 1$,

$$\begin{aligned} A_{1,0} &:= A_0, \\ A_{1,1} &= F_{\lambda_1}^*(A_{1,0}) = F_{\lambda_1}^*(A_0), \\ A_{1,2} &= F_{\lambda_1}^*(A_{1,1}) = {F^*}_{\lambda_1}^2(A_0), \\ &\vdots \\ A_{1,n_1} &= {F^*}_{\lambda_1}^{n_1}(A_0), \end{aligned}$$

where

$$\begin{aligned} d_H(A_{1,k}, A(\lambda_1)) &\leq \frac{L^k}{1-L} d_H(A_0, F_{\lambda_1}^*(A_0)) = \\ &= \frac{L^k}{1-L} d_H(A_{1,0}, A_{1,1}). \end{aligned}$$

It is clear that there exists $n_1 \in \mathbb{N}$ such that $d_H(A_{1,n_1}, A(\lambda_1)) \leq r$.

For $j = 2, \ldots, m-1$, we use an analogous procedure.

For $j = m$,

$$\begin{aligned} A_{m,0} &:= A_{m-1,n_{m-1}}, \\ A_{m,1} &= F_{\lambda_m}^*(A_{m,0}), \\ &\vdots \\ A_{m,n_m} &= {F^*}_{\lambda_m}^{n_m}(A_{m,0}) = \\ &= {F^*}_{\lambda_m}^{n_m}({F^*}_{\lambda_{m-1}}^{n_{m-1}}(\cdots {F^*}_{\lambda_1}^{n_1}(A_0)\cdots)), \end{aligned}$$

where

$$d_H(A_{m,n_m}, A(\lambda_m)) = d_H(A_{m,n_m}, A(1)) \leq \varepsilon.$$

Thus, $\widetilde{A}_1 := A_{m,n_m}$ is an ε-approximation of $A(1)$.

As we can see, the computational part of Theorem 2 is based on iterations of contractions. The remaining part of the paper is devoted to this topic with the focus on (numerical) inaccuracy.

2. Approximation of multivalued fractals

As attractive fixed-poits of the Hutchinson-Barnsley operators of the IMS of contractions with compact values, the multivalued fractals can be approximated by iterates of any compact set w.r.t. the Hutchinson-Barnsley operator. Due to the nonaccurate character of numerical calculations, the IMS of contractions with compact values will be alternatively studied, when applying the Shadowing Lemma in metric spaces ([Žáč92], cf. also [Bie99]).

The existence of an accurate orbit in the proximity of a pseudo-orbit is important for validity of numerical simulations. Numerically computed orbits are in fact pseudo-orbits and, therefore, the problem of their being in a neighbourhood of a real orbit for a sufficiently long time arises, i.e. the question whether the numerical calculation has a real meaning.

Hence, let (X, d) be a metric space, $f : X \to X$ be a map, and δ and ε be positive reals.

A sequence $\{x_k\}_{k=0}^{\infty} \subset X$ is called a *δ-pseudo-orbit* of the map f if

$$d(f(x_k), x_{k+1}) \leq \delta, \quad k = 0, 1, 2, \ldots. \tag{9}$$

We say that the δ-pseudo-orbit $\{x_k\}_{k=0}^{\infty}$ is *ε-shadowed* by some real orbit of f if there exists $y \in X$ such that

$$d(f^k(y), x_k) \leq \varepsilon, \quad k = 0, 1, 2, \ldots. \tag{10}$$

The shadowing property for the IMS of contractions with compact values is stated in the following proposition (which is a weaker form of that for weak contractions in [AFGL])

Proposition 1 (cf. [AFGL]). *Let $F^* : \mathcal{K}(X) \to \mathcal{K}(X)$ be the Hutchinson-Barnsley operator of the IMS $\{\varphi_i : X \multimap X, i = 1, \ldots, n\}$ of contractions with compact values.*

Then, for every $\varepsilon > 0$, there exist $\mu \in (0, \varepsilon)$ and $\delta > 0$ such that every δ-pseudo-orbit $\{A_k\}_{k=0}^{\infty}$ is ε-shadowed by the orbit $\{{F^*}^k(C)\}$, where $C \in \mathcal{K}(X)$ satisfies $d_H(A_0, C) \leq \mu$.

For numerical δ-pseudo-orbits, δ means the upper estimate of the error in each step, and so, it is not generally arbitrarily small positive, but there exists a lower bound $\delta_0 > 0$ for the accuracy of calculations.

Thus, the theoretical shadowing result ($\forall \varepsilon > 0 \ \exists \delta > 0$) needs to be modified to the form ($\forall \varepsilon = \varepsilon(\delta_0) > 0 \ \exists \delta > \delta_0 > 0$), looking only for such situations where we are able to describe the relations among ε, δ_0 and δ. As pointed out in the following Proposition 2, this is possible for contractions, where we place emphasis on approximation of related fixed-points.

2.1. **Approximation of fixed-points of contractions.** Obviously, every orbit of a contraction tends to the unique fixed-point. However, due to inaccuracy of numerical calculations, we cannot reach this point with an arbitrary precision in a real situation.

Proposition 2 (cf. [AFGL]). *Let $\{\varphi_i : X \multimap X; i = 1,\dots,n\}$ be a system of contractions with compact values on a complete metric space (X,d) with Lipschitz constants $L_i < 1$, $i = 1,\dots,n$, and with a multivalued fractal (fixed-point of F^*) A^*. Let $\varepsilon > 0$, $\delta > 0$ and $L := \max\{L_i \mid i = 1,\dots,n\}$ satisfy the inequality*

$$\delta < \varepsilon(1-L). \tag{11}$$

Then, for an arbitrary δ-pseudo-orbit $\{A_k\}_{k=0}^{\infty}$ of the (Hutchinson-Barnsley operator) F^*, we have

$$d_H(A^*, A_m)) \le \varepsilon, \tag{12}$$

whenever

$$(0 \le)\ m \ge \ln\left(\frac{\varepsilon(1-L)-\delta}{d_H(A_0,A_1)}\right)\frac{1}{\ln L}. \tag{13}$$

(In particular, for $d_H(A_0,A_1) \in [0, \varepsilon(1-L)-\delta)$, the right hand side of the inequality 13 is negative, and so we can obtain $m = 0$, i.e. $d_H(A^*, A_0) \le \varepsilon$.)

Proof. From the definition of δ-pseudo-orbit:

$$d_H(F^*(A_k), A_{k+1}) \le \delta, \quad k \in \mathbb{N}.$$

Thus

$$\begin{array}{rcl}
d_H(F^*(A_0), A_1) & \le & \delta, \\
d_H({F^*}^2(A_0), A_2) & \le & d_H(F^*(F^*(A_0)), F^*(A_1)) + d_H(F^*(A_1), A_2) \le \\
 & \le & L d_H(F^*(A_0), A_1) + \delta \le L\delta + \delta, \\
 & \vdots & \\
d_H({F^*}^k(A_0), A_k) & \le & L^{k-1}\delta + \dots + L^2\delta + L\delta + \delta = \dfrac{1-L^k}{1-L}\delta, \\
 & \downarrow k \to \infty & \\
d_H(A^*, A_\infty) & \le & \dfrac{\delta}{1-L}.
\end{array}$$

From the last inequality we can obtain the lower bound for ε in (12) for possible reasonable ε-approximation of A^* by some member of $\{A_k\}_{k=0}^{\infty}$:

$$d_H(A^*, A_\infty) \le \frac{\delta}{1-L} \le \varepsilon.$$

Inequality (13) can be obtained as follows. We want to get an ε-approximation of A^* by some $A_m \in \{A_k\}_{k=0}^{\infty}$, i.e.

$$d_H(A^*, A_m) \le \varepsilon,$$

$$\begin{array}{rcl}
d_H(A^*, A_m) & \le & d_H(A^*, {F^*}^m(A_0)) + d_H({F^*}^m(A_0), A_m) \le \\
 & \le & \dfrac{L^m}{1-L} d_H(A_0, F^*(A_0)) + \dfrac{1-L^m}{1-L}\delta \le \\
 & \le & \dfrac{L^m}{1-L}(d_H(A_0, A_1) + d_H(A_1, F^*(A_0))) + \dfrac{1-L^m}{1-L}\delta \le \\
 & \le & \dfrac{L^m}{1-L}(d_H(A_0, A_1) + \delta) + \dfrac{1-L^m}{1-L}\delta = \\
 & = & \dfrac{L^m d_H(A_0, A_1) + \delta}{1-L} \le \varepsilon,
\end{array}$$

and so, for $d_H(A_0, A_1) > 0$,

$$m \geq \ln\left(\frac{\varepsilon(1-L)-\delta}{d_H(A_0,A_1)}\right)\frac{1}{\ln L}.$$

In the case $d_H(A_0, A_1) = 0$ we can set $m := 0$, because

$$d_H(A^*, A_m) \quad \leq \quad \frac{\delta}{1-L} \leq \varepsilon, \text{ for all } m \geq 0,$$

and so already A_0 is ε-approximation of A^*, i.e. $d_H(A^*, A_0) \leq \varepsilon$. □

Remark 1. It is easy to see (from the proof of Proposition 2) that every such δ-pseudo-orbit $\{A_k\}_{k=0}^{\infty}$ is ε-shadowed (or, more precisely, $\frac{\delta}{1-L}$-shadowed) by any orbit $\{F^*(A)\}$, where $A \in \mathcal{K}(X)$ satisfies $d_H(A_0, A) \leq \delta$.

Remark 2. In the real situation, (11) could be completed by the lower bound for δ:

$$\delta_0 \leq \delta < \varepsilon(1-L).$$

Remark 3. As observed in [GG04], the application of the Shadowing Lemma is limited just by the IMS of contractions.

3. Stochastic approximation

In [LM94] one can find a result, which is related to the approximation of fractals of iterated function systems by random iterations of generating contractions. We show in Proposition 3 that it is (together with Proposition 2) applicable to multivalued fractals of certain class of iterated multifunction systems.

Proposition 3. *Let*

$$\{\varphi_i : X \to \mathcal{K}(X) \mid i = 1, \dots, n\}, \tag{14}$$

be an iterated multifunction system of contractions with compact values and concractivity factors $L_i \in [0,1)$, $i = 1, \dots, n$, F^* its Hutchinson-Barnsley operator, and A^* its multivalued fractal.

Let there exist an iterated function system of contractions

$$\{f_i : X \to X \mid i = 1, \dots, m\}, \quad \hat{L}_i \in [0,1), \tag{15}$$

which δ_1-approximates the IMS (14), i.e.

$$d_H\left(F^*(A), \hat{F}^*(A)\right) \leq \delta_1, \quad A \in \mathcal{K}(X),$$

where $\hat{F}^*$ is the Hutchinson-Barnsley operator of (15).

Let

$$p_1, \dots, p_N, \qquad p_i > 0, \quad i = 1, \dots, n, \quad \sum_{i \in I} p_i = 1,$$

be a probabilistic vector and $\{\xi_k\}_{k=0}^{\infty}$ a sequence of independent random variables such that

$$\mathrm{prob}(\xi_n = i) = p_i, \quad \textit{for } i = 1, \dots, n.$$

For IFS with probabilities $\{(f_i, p_i), i = 1, \dots, n\}$, consider its orbit $\{x_k\}_{k=0}^{\infty}$ with an initial point x_0 and

$$x_{k+1} = f_{\xi_k}(x_k), \quad k = 0, 1, \dots.$$

Then for every $x_0 \in X$ *and* $\varepsilon > 0$ *there exist* $k_0 = k_0(\varepsilon)$ *and* $j_0 = j_0(\varepsilon)$ *such that*

$$\operatorname{prob}(d_H(\{x_k, \dots, x_{k+j}\}, A^*) < \frac{\delta_1}{1-L} + \varepsilon) > 1 - \varepsilon, \tag{16}$$

for $k \geq k_0$*,* $j \geq j_0$*, and* $L = \max\{L_i \mid i = 1, \dots, n\}$*.*

It says the following. If we cancel the first k_0 or more elements of the orbit $\{x_k\}_{k=0}^{\infty}$, then the probability that a sufficiently long segment $x_k, \dots, x_{k+j}$ approximates A^* with accuracy $\frac{\delta_1}{1-L} + \varepsilon$ is greater than $1 - \varepsilon$.

Proof. We denote by $\hat{A}^*$ the fractal of (15).

The trivial case when (14) is a system of single-valued contractions (here we can take (15)=(14) and so $\delta_1 = 0$) is presented in [LM94].

For the case of a nontrivial (14), using Proposition 2, we can obtain the following inequality

$$d_H(A^*, \hat{A}^*) \leq \frac{\delta_1}{1-L},$$

which already gives (16). □

Our aim is to complete this result by the (numerical) inaccuracy part in the following

Theorem 3. *Let the assumptions of Proposition 3 be fulfilled.*

Let $\{\tilde{x}_k\}_{k=0}^{\infty}$ *be a* δ_2*-pseudo-orbit of (15) with an initial point* $\tilde{x}_0$*, i.e., for all* $k \geq 0$*, there exists* $i \in \{1, \dots, m\}$*, such that*

$$d(\bar{x}_{k+1}, f_i(\bar{x}_k)) \leq \delta_2. \tag{17}$$

Then for every $\tilde{x}_0 \in X$ *and* $\varepsilon > 0$ *there exist* $k_0 = k_0(\varepsilon)$ *and* $j_0 = j_0(\varepsilon)$ *such that*

$$\operatorname{prob}(d_H(\{\tilde{x}_k, \dots, \tilde{x}_{k+j}\}, A^*) < \frac{\delta_1}{1-L} + \frac{\delta_2}{1-\hat{L}} + \varepsilon) > 1 - \varepsilon, \tag{18}$$

for $k \geq k_0$*,* $j \geq j_0$*,* $L = \max\{L_i \mid i = 1, \dots, n\}$ *and* $\hat{L} = \max\left\{\hat{L}_i \mid i = 1, \dots, m\right\}$*.*

Proof. From the definition of δ_2-pseudo-orbit $\{\tilde{x}_k\}_{k=0}^{\infty}$ of (15), it is easy to see that we can construct, for arbitrary $x_0 \in X$, $d(x_0, \tilde{x}_0) \leq \delta_2$, the orbit $\{x_k\}_{k=0}^{\infty}$ of (15) such that $x_{k+1} = f_i(x_k)$, choosing, for every $k \geq 0$, the same function as in (17). Using Proposition 2 (and Remark 1) we can obtain the following two inequalities

$$d(x_k, \tilde{x}_k) \leq \frac{1-\hat{L}^k}{1-\hat{L}}\delta_2 \leq \frac{\delta_2}{1-\hat{L}},$$

$$d_H(\{x_k\}_{k=0}^{\infty}, \{\tilde{x}_k\}_{k=0}^{\infty}) \leq \frac{\delta_2}{1-\hat{L}}.$$

Thus, (cf. 18)

$$d_H(\{x_k, \dots, x_{k+j}\}, \{\tilde{x}_k, \dots, \tilde{x}_{k+j}\}) \leq \frac{\delta_2}{1-\hat{L}}.$$

The last inequality together with (16) in Proposition 3 gives (18). □

Now, Proposition 2 and Theorem 3 will be applied, in illustrating Examples 1 and 2, to a trivial system of one single-valued contraction and to an IMS.

Example 1 (Shadowing of a single-valued contraction)**.** Consider map (see Fig.1)

$$f(x) = \frac{3}{4}x, \quad x \in \mathbb{R}.$$

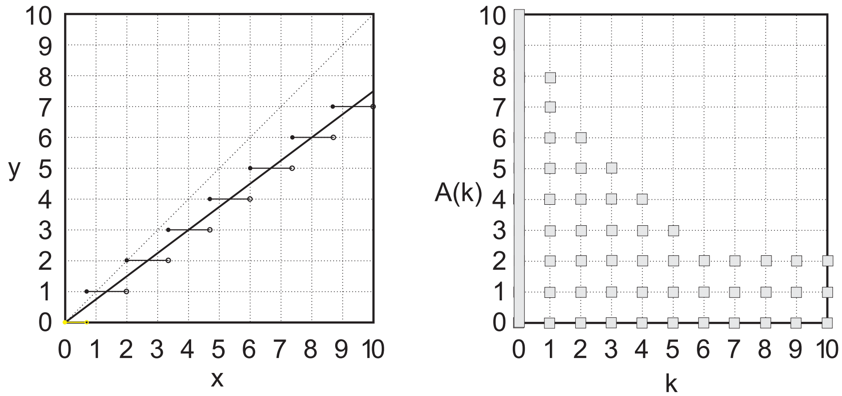

FIGURE 1. Map $f(x) = \frac{3}{4}x$, digitization $\tilde{f}$, and iterations w.r.t. $\tilde{f}$

The induced map $f^* : \mathcal{K}(\mathbb{R}) \to \mathcal{K}(\mathbb{R})$ takes the form

$$f^*(A) = \bigcup_{x \in A} \frac{3}{4}x, \quad A \in \mathcal{K}(\mathbb{R}).$$

f is obviously a contraction with constant $\frac{3}{4}$, and subsequently (cf. [AF04])

$$d_H(f^*(A), f^*(B)) \leq \frac{3}{4} d_H(A, B), \qquad A, B \in \mathcal{K}(\mathbb{R}).$$

Thus, f^* is a contraction with constant $\frac{3}{4}$, too.

Now, let $\{A_n\}_{n=0}^{\infty}$ be an orbit of f^* in $\mathcal{K}(\mathbb{R})$ with $A_0 \in \mathcal{K}(\mathbb{R})$ and $A_n = {f^*}^n(A_0)$. Since f^* is a contraction in a complete metric space $(\mathcal{K}(\mathbb{R}), d_H)$, A_n tends w.r.t. d_H to a unique fixed-point $(\mathcal{K}(\mathbb{R}) \ni) A^* = \{0\}$.

However, in a real situation, to reach the fixed-point A^* numerically, we are restricted by limited accuracy and finite number of iterations. Inequality (13) in Lemma 2 says how many iterations m are sufficient in order to approximate A^* with given accuracy ε. Number δ relates to the accuracy of a single iteration and $d_H(A_0, A_1)$ stands for the distance between the original and the first iteration, provided (11), i.e. $\delta < \varepsilon(1 - L)$, where L is the contraction constant. L is related to the given map, while δ depends on the technical possibilities of computation. Inequality (11) thus relates to the highest accuracy of ε.

Since we can only compute with rational numbers with limited number of decimal places, let us only compute, for the sake of simplicity, with no decimal place, i.e. with step 1. Considering such a mesh on $\mathbb{R}$, it can be easily checked that any subset of $\mathbb{R}$ can be approximated by the subsets of the mesh with the accuracy $\delta = \frac{1}{2}$ in the Hausdorff metric. Obviously, for our purpose, only compact subsets of $\mathbb{R}$ will be taken into account.

Hence, working on such a $\frac{1}{2}$-mesh, every value of f^* will be approximated with the accuracy of $\delta = \frac{1}{2}$. For $L = \frac{3}{4}$, one can make an estimate for ε in (11), namely

$\frac{1}{2} < \varepsilon(1-\frac{3}{4})$, i.e. $\varepsilon > 2$, which refers to the unreachable limit of 2. Since only integer multiples of δ are meaningful for the required accuracy, let us take $\varepsilon = 5\frac{1}{2}$.

Denoting by $\tilde{f}$ the digitization of f (see Fig. 1), its first iteration, for $A_0 = \{10\}$, is $A_1 = \{8\}$, where $d_H(A_0, A_1) = 2$.

Hence, substituting for f^* into formula (13), we get

$$m \geq \ln\left(\frac{\frac{5}{2}(1-\frac{3}{4})-\frac{1}{2}}{2}\right)\frac{1}{\ln\frac{3}{4}} \doteq 9.6.$$

For $m \geq 10$, we have thus guaranteed the ε-accuracy ($\varepsilon = \frac{5}{2}$) to approximate $A^* = \{0\}$. By practical computation, we obtain orbit $\{A_n\}_{n\geq 0}$ as follows:

$$A_0 = \{10\}\,,\ A_1 = \{8\}\,,\ \ldots,\ A_5 = \{3\}\,,\ A_6 = \{2\}\,,\ A_k = A_6,\ k \geq 7,$$

where the Hausdorff distance $d_H(A_6, A^*) = 2$ is indeed less than $\varepsilon = \frac{5}{2}$.

Analogously, we present the computations for $A_0 = [0, 10]$ (interval). Here, $A_1 = \{0, \ldots, 8\}$, and so the estimate for m is the same, i.e. $m \geq 10$. The iterations (see Fig.1):

$$A_0 = [0, 10],\ \ldots,\ A_5 = \{0, \ldots, 3\}\,,\ A_6 = \{0, \ldots, 2\}\,,\ A_k = A_6,\ k \geq 7,$$

where the Hausdorff distance $d_H(A_6, A^*)$ is 2, too. Thus, even for a single-valued contraction, we obtain multivalued $\frac{5}{2}$-approximation $\{0, 1, 2\}$ of the fixed-point $\{0\}$, iterating the digitization.

Example 2 (IMS of contractions: shadowing and stochastic generating)**.** Consider the following iterated multifunction system on the unite square $X = [0,1]\times[0,1] \subset \mathbb{R}^2$:

$$\begin{aligned}
f_1\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix},\\
f_2\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix},\\
f_3\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix},\\
f_4\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.\\
\varphi\begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{BBU},
\end{aligned} \tag{19}$$

where $\varphi : X \to \mathcal{K}(X)$ is a constant multivalued map, which maps every point to letters **BBU**, positioned in the center of X. IMS (19) consists of a multivalued contraction and of four single-valued contractions.

In order to apply Lemma 2, we have to find the constants in inequality (13). Following the arguments in the foregoing Example 1, we can take $L = \frac{1}{3}$, and $\delta = \frac{2}{100}$, for two decimal places computing, and for the Manhattan metric, i.e.

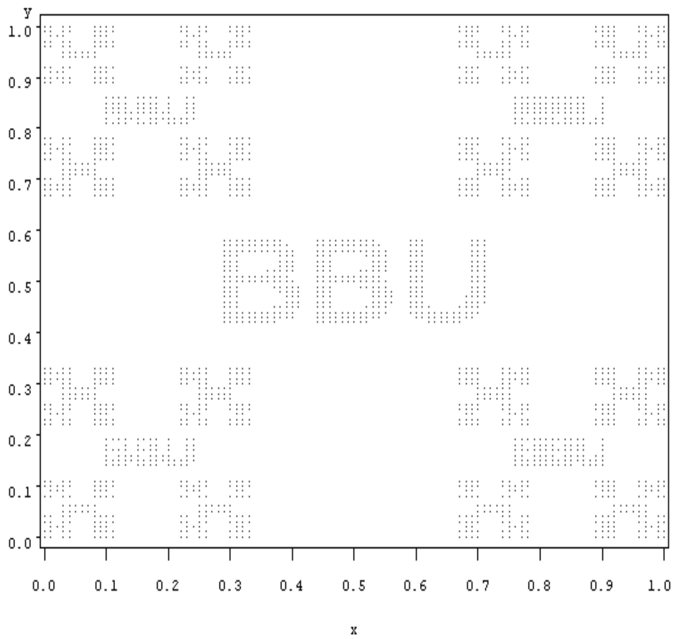

FIGURE 2. $\frac{4}{100}$-approximation of fractal of IMS (19)

$d((x_1,x_2),(y_1,y_2)) = |y_1 - x_1| + |y_2 - x_2|$. Then, for $\varepsilon = 0.1$ and $A_0 = \{(0,0)\}$, we have

$$A_1 = \left\{ \mathbf{BBU}, (0,0), \left(\frac{2}{3}, 0\right), \left(0, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}\right) \right\},$$

$d_H(A_0, A_1) = \left|\frac{2}{3} - 0\right| + \left|\frac{2}{3} - 0\right| = \frac{4}{3}$, and so

$$m \geq \ln \left(\frac{\frac{4}{100}\left(1 - \frac{1}{3}\right) - \frac{2}{100}}{\frac{4}{3}} \right) \frac{1}{\ln \frac{1}{3}} \doteq 4.8.$$

Thus, the sufficient number of iterations is $m = 5$ (see Fig.2).

Furthermore, the stochastically generated fractal of IMS (19) in Fig.3 is based on application of Theorem 3, where $\delta_1 = \delta_2 = \frac{1}{100}$, $L = \hat{L} = \frac{1}{3}$ and $\frac{\delta_1}{1-L} + \frac{\delta_2}{1-\hat{L}} = 2\frac{1}{100}\frac{3}{2} = \frac{3}{100}$. Thus, we obtain stochastical $\frac{3}{2}$-approximation.

Finally, we conclude these approximations by a graphically obtained multivalued fractal in Figure 4.

REFERENCES

[AF04] J. Andres and J. Fišer, *Metric and topological multivalued fractals*, Int. J. Bifurc. Chaos **14** (2004), no. 4, 1277-1289.

[AFGL] J. Andres, J. Fišer, G. Gabor, and K. Leśniak, *Multivalued fractals*, Chaos, Solitons and Fractals, to appear.

[AG01] J. Andres and L. Górniewicz, *On the Banach contraction principle for multivalued mappings*, In: Approximation, Optimization and Mathematical Economics. Proceedings of the 5th international conference on "Approximation and Optimization in the Caribbean",

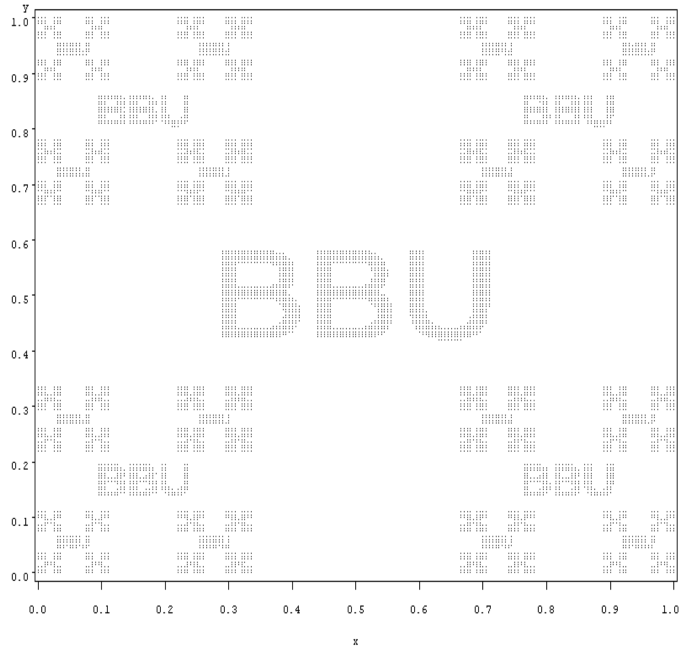

Figure 3. Stochactically generated $\frac{3}{2}$-approximation of fractal of IMS (19)

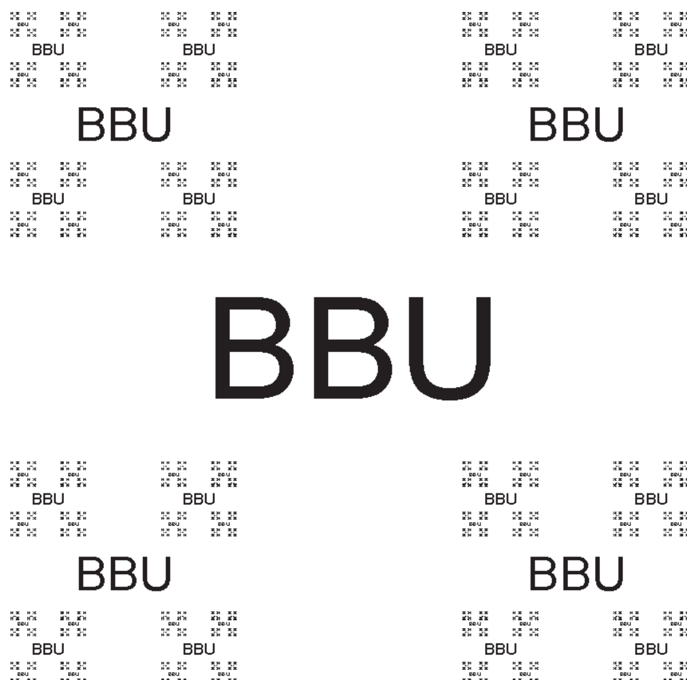

Figure 4. Graphically generated fractal of IMS (19)

Guadeloupe, French West Indies, March 29 - April 2, 1999 (M. Lassonde, ed.), Heidelberg: Physica-Verlag. 1-23 , 2001.

[AG03] ———, *Topological Fixed Point Principles for Boundary Value Problems*, Serie: Topological Fixed Point Theory and Its Applications, Kluwer, Dordrecht, 2003.

[And] J. Andres, *Continuation principles for fractals*, Submitted.

[Bar88] M. F. Barnsley, *Fractals Everywhere*, Academic Press, New York, 1988.

[Ber02] Vasile Berinde, *Iterative approximation of fixed points*, Baia Mare: Efemeride. xii, 2002.

[Bie99] A. Bielecky, *Approximation of attractors by pseudotrajectories of iterated function systems*, Univ. Iagel. Acta Math. **32** (1999), 173–179.

[Eda95] A. Edalat, *Dynamical systems, measures and fractals via domain theory*, Inform. and Comput. **120** (1995), no. 1, 32–48.

[EP] R. Espínola and A. Petruşel, *Existence and data dependence of fixed points for multivalued operators on gauge spaces*, Preprint (2004).

[FG94] M. Frigon and A. Granas, *Résultats du type de Leray-Schauder pour des contractions multivoques*, Topol. Math. Nonlin. Anal. **4** (1994), 197–208.

[GD03] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, Berlin, 2003.

[GG04] V. Glavan and V. Guţu, 2004, Lectures given at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

[Gra72] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANR's*, Bull. Soc. Math. France **100** (1972), 209–228.

[Gra94] ———, *Continuation method for contractive maps*, Topol. Math. Nonlin. Anal. **3** (1994), 375–379.

[Hig96] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, SIAM Publ., Philadelphia, PA, 1996.

[Hut81] J. E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[Jac96] J. Jachymski, *Continuous dependence of attractors of iterated function systems*, J. Math. Anal. Appl. **198** (1996), 221–226.

[Jac98] ———, *Fixed point theorems in metric and uniform spaces via the Knaster-Tarski principle*, Nonlin. Anal., T.M.A. **32** (1998), no. 2, 225–233.

[JGP00] J. Jachymski, L. Gajek, and P. Pokarowski, *The Tarski-Kantorovitch principle and the theory of iterated function systems*, Bull. Austral. Math. Soc. **61** (2000), 247–261.

[Kie02] B. Kieninger, *Iterated Function Systems on Compact Hausdorff Spaces*, Ph.D. thesis, Institut für Mathematik der naturwissenschaftlichen Fakultät der Universität Augsburg, Augsburg 2002.

[LM94] A. Lasota and M. C. Mackey, *Chaos, fractals and noise-stochastic aspects of dynamics*, Springer Verlag, New York, 1994.

[LM00] A. Lasota and J. Myjak, *Attractors of multifunctions*, Bull. Pol. Ac.: Math. **48** (2000), no. 3, 319–334.

[Pet01] A. Petruşel, *Single-valued and multi-valued Meir-Keeler type operators*, Rev. d'Anal. Num. et de Thèorie de l'Approx. **30** (2001), no. 1, 75–80.

[Pet02] ———, *Fixed point theory with applications to dynamical systems and fractals*, Semin. Fixed Point Theory Cluj-Napoca **3** (2002), 305–316.

[PR] A. Petruşel and I. A. Rus, *Multivalued Picard and weakly Picard operators*, Preprint (2004).

[PR01] ———, *Dynamics on $(P_{cp}(X), H_d)$ generated by a finite family of multi-valued operators on (X, d)*, Math. Moravica **5** (2001), 103–110.

[Pre02] R. Precup, *Continuation results for mappings of contractive type*, Semin. Fixed Point Theory Cluj-Napoca **2** (2002), 23–40.

[Rho01] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Analysis **47** (2001), 2683–2693.

[Rus01] Ioan A. Rus, *Generalized contractions and applications*, Cluj-Napoca: Cluj University Press, 2001.

[SAV84] J. Soto-Andrade and F.J. Varela, *Self-Reference and Fixed Points: A Discussion and an Extension of Lawvere's Theorem*, Acta Appl. Math. **2** (1984), 1–19.

[Tod76] M. J. Todd, *The Computation of Fixed Points and Applications*, Springer, Berlin, 1976.

[Žáč92] T. Žáčik, *On a shadowing lemma in metric spaces*, Mathematica Bohemica **117** (1992), no. 2, 137–149.

Faculty of Science
Department of Mathematical Analysis and Applications of Mathematics

This preprint series was founded in 1997. Its purpose is to present manuscripts of submitted or unpublished papers and reports of members of the Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University in Olomouc. The authors are fully responsible for the content of the preprints. Mail address:

Department of Mathematical Analysis
Faculty of Science
Palacký University
Tomkova 40, Hejčín
779 00 Olomouc
Czech Republic

Tel. (+420) 585 412 210, 412 604, 411 646, 411 647

Latest preprints of the series:

12/2004 *Jiří Fišer:* Numerical aspects of multivalued fractals.
11/2004 *Lubomír Kubáček:* Some remarks to multivariate regression model.
10/2004 *Jaroslav Marek:* Estimation in connecting measurements with constraints of type II.
9/2004 *Lubomír Kubáček:* Nonlinear problems of equivalent adjustment algorithms.
8/2004 *Lubomír Kubáček:* Underparametrization in a regression model with constraints.
7/2004 *Lubomír Kubáček, Eva Tesaříková:* Tests in weakly nonlinear regression model.
6/2004 *Lubomír Kubáček, Jaroslav Marek:* Partial optimum estimator in two stage regression model with constraints and a problem of equivalence.
5/2004 *Jan Andres:* Continuation principles for fractals.
4/2004 *Jan Andres:* Nielsen number and differential equations.
3/2004 *Eva Fišerová:* One statistical problem in railway engineering.
2/2004 *Jan Andres, Jiří Fišer, Grzegorz Gabor, Krzysztof Leśniak:* Multivalued fractals.
1/2004 *Lubomír Kubáček, Eva Tesaříková:* Estimation of dispersion in nonlinear regression models with constraints.